How to Draw a Picture of an Unknown Inclusion from Boundary Measurements.
Two Mathematical Inversion Algorithms

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Abstract

We report two new mathematical inversion algorithms for the electric impedance tomography. An application to the reconstruction problem of the unknown boundary on which the Neumann derivative of the solution of the Helmholtz equation vanishes is included.

1 Introduction

In this paper we present two mathematical inversion algorithms for the electric impedance tomography, which should take a place in [2]. These materials are mainly taken from [4], [8] and [9]. There are, however, some new results not included therein.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with connected smooth boundary. We consider \( \Omega \) an isotropic electric conductive body and denote its electric conductivity by \( \gamma \), which is a real-valued function on \( \Omega \). We assume that \( \gamma \) is in \( L^\infty(\Omega) \) and uniformly positive. Following Calderón’s formulation, we prescribe the voltage potential \( f \in H^{1/2}(\partial \Omega) \). Then the voltage potential inside \( \Omega \) is the unique weak solution \( u \in H^1(\Omega) \) of the Dirichlet problem

\[
\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \quad u = f \quad \text{on } \partial \Omega.
\]

We define the Dirichlet-to-Neumann map \( \Lambda_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) by the formula

\[
< \Lambda_\gamma(f), g > = \int_\Omega \gamma \nabla u \cdot \nabla v,
\]

where \( g \in H^{1/2}(\partial \Omega) \) and \( v \) is any \( H^1(\Omega) \) function such that \( v = g \) on \( \partial \Omega \). \( \Lambda_\gamma f \) is the electric current distribution applied to \( \partial \Omega \) which produces \( f \). In principle these quantities can be measured by means of a combination of a voltmeter and amperemeter at the boundary.

The problem is to find a reconstruction procedure of \( \gamma \) by means of the Dirichlet-to-Neumann map \( \Lambda_\gamma \). It should be noted that we seek only an exact reconstruction procedure. It means that from such procedure one can automatically get a uniqueness of the conductivity having the same Dirichlet-to-Neumann map or its restriction to a set of
common Dirichlet data. We refer the reader to [2] for a list of non exact reconstruction procedures. When \( \gamma \) has some regularity, say \( \gamma \in W^{2,\infty}(\Omega) \), we know from Nachman’s result of [13](see also [14]) or Novikov’s result of [15](see also [16]) that

**Theorem 1.1.** Assume that \( \mathbb{R}^3 \setminus \overline{\Omega} \) is connected. There is a reconstruction procedure of \( \gamma \) from the full knowledge of \( \Lambda_{\gamma} \).

They found a characterization of the trace of the special solutions on \( \partial \Omega \) by means of the Dirichlet-to-Neumann map which was constructed by Sylvester and Uhlmann in [18]. Unfortunately, one can not apply their results to the case where \( \gamma \) has an unknown surface of discontinuity because of the strong singularity of \( \gamma \). When \( \gamma \) is piecewise real analytic, Kohn and Vogelius [11] proved a uniqueness theorem. Alessandrini [1] also proved a uniqueness theorem in a more general situation. However their proofs don’t contain any reconstruction procedure.

Isakov [10] was the first who considered the case when the conductivity is known outside a subset \( D \) of \( \Omega \), unknown inside \( D \) and not necessarily real analytic therein. He proved a uniqueness of the shape of \( D \) and the conductivity inside \( D \). Ikehata [6] found a simple proof of the uniqueness of the shape of \( D \) and pointed out that it is not necessary for the proof to assume any regularity of the conductivity inside \( D \). The proof is based on a system of integral inequalities established in [3]. But both also don’t contain any reconstruction procedure.

The purpose of this paper is to report two types of the reconstruction procedures presented in [4],[8] and [9] in that case.

We formulate our problem more precisely. We assume that \( \gamma \) takes the form

\[
\gamma = \gamma_0 + \chi_D h
\]

where \( D \subset \Omega \), \( \gamma_0 \) is a known reference conductivity, \( h \in L^\infty(D) \), \( h \) satisfies \( \gamma_0(x) + h(x) \geq C \) for almost all \( x \in D \) and a jump condition mentioned in Sections 2 and 3.

For simplicity, in this paper, we take \( \gamma_0 \equiv 1 \). We call the pair \((D, h)\) an inclusion embedded in \( \Omega \). Now our problem is stated as below.

**Inverse Problem A.** Assume that \( D \) and \( h \) are unknown. Find a reconstruction procedure of the shape of \( D \) and the value of \( h \) on \( D \) by means of the Dirichlet-to-Neumann map.

In Section 2 we give a reconstruction procedure of \( D \) and \( h \) which is based on the Runge approximation property and a fundamental solution of the Laplacian. We call this method the probe method (see [8]). In Section 3 we describe a reconstruction procedure of the convex hull of \( D \) based on the complex exponentially growing solution used by Calderón. The method is quite different from Calderón’s one.

Sections 4 and 5 are devoted to an application of the idea of the second algorithm to a reconstruction problem of the boundary of an unknown object on which the Neumann derivative of a solution of the governing equation vanishes. These are new and not included in [9].

Notice that the idea of the first algorithm was applied to the inverse scattering problem at a fixed frequency in [7] and [5].
2 The First Algorithm (The Probe Method)

In this section, we assume that $D$ is open, $\overline{D} \subset \Omega$, $\partial D$ is Lipschitz and that $\Omega \setminus \overline{D}$ is connected.

The jump condition is the following assumption:

$$\forall a \in \partial D \exists C > 0 \exists \delta > 0$$

$$h(x) \geq C \text{ in } B(a, \delta) \cap D$$

or

$$-h(x) \geq C \text{ in } B(a, \delta) \cap D.$$ 

Here we denoted the open ball with radius $\delta$ centered at $a$ by $B(a, \delta)$.

Under these assumptions we will show how to reconstruct $D$ and $h$. First we take a continuous curve $c : [0, 1] \rightarrow \Omega$ satisfying $c(0), c(1) \in \partial \Omega$ and $c(t) \in \Omega$ for $0 < t < 1$. We call this curve a needle. It plays a central role in this section. The function

$$G(x) = \frac{1}{4\pi|x|}, \; x \in \mathbb{R}^3$$

is the standard fundamental solution of the Laplacian. We denote by $\Gamma$ a nonempty open subset of $\partial \Omega$. We specify

**Dirichlet Data.** For each $t \in ]0, 1[$, take a sequence of $H^1(\Omega)$ harmonic functions $(v_n)$ in such a way that

$$\operatorname{supp} (v_n|_{\partial \Omega}) \subset \Gamma$$

$$v_n \rightarrow G(\cdot - c(t)) \text{ in } H^1_{\text{loc}}(\Omega \setminus \{c(t)\mid 0 \leq t' \leq t\}).$$

Take the trace of each $v_n$ on $\partial \Omega$. To indicate its dependence on $c$, we write

$$v_n|_{\partial \Omega} = f_n(\cdot ; c(t)).$$

This is our Dirichlet data.

**Remark.** The existence of such sequence is a consequence of the Runge approximation property of the equation $\Delta u = 0$(see [4], [8] and [10]).

Using the Dirichlet data, we define

**Observation Data.** For each $0 < t < 1$ set

$$I(t, c) = \lim_{n \rightarrow \infty} (\Lambda_\gamma - \Lambda_1)f_n(\cdot ; c(t)), f_n(\cdot ; c(t)) >$$

if it exists. We define the set $T(c)$:

$$T(c) = \{0 < t < 1 \mid I(s, c) \text{ exists for all } 0 < s < t \text{ and } \sup_{0 < s < t} |I(s, c)| < \infty\}.$$ 

The definition of $T(c)$ is slightly different from that of [4] but everything goes well.

We recall

**Impact Parameter.** Define

$$t(c; D) = \sup \{0 < t < 1 \mid c(s) \in \Omega \setminus \overline{D} \text{ for all } 0 < s < t\}.$$
We know that $t(c; D) = 1$ if $c$ never touches $\bar{D}$ and $t(c; D) < 1$ if $c$ touches $\bar{D}$ and $t(c; D)$ is the first hitting time.

The result is

**Theorem 2.1** ([4], [8]). For any needle $c$

$$T(c) = ]0, t(c; D)[$$

and thus the formula

$$\sup T(c) = t(c; D)$$

(1)

is valid.

A combination of the formula

$$\partial D = \{c(t)|_{t=t(c; D)} | t(c; D) < 1\}$$

and (1) gives a reconstruction formula of $\partial D$.

The proof is divided into two parts. The first is to prove the existence of $I(t, c)$ for $t \in ]0, t(c; D)[$. The second and the key is to prove

$$|I(t, c)| \approx \int_D |\nabla G(x - c(t))|^2 dx$$

as $t \uparrow t(c; D)$.

In [8] the same result is proved when $\gamma_0 \in C^{0,1}(\Omega)$. In [4] we applied this idea also to a special version of the inverse problem treated by Sun and Uhlmann [19]. An application to an inverse problem for the system of the equations in elasticity is included in [8].

The next problem is how to reconstruct $h$. We will briefly describe it. For the exact statement, which doesn’t require the existence of the Green’s function, see [8]. Let $G_\gamma(x, y)$ be the Green’s function for the operator $\nabla \cdot \gamma \nabla$ in $\Omega$. $x$ and $y$ denote the receiver point and the source point, respectively.

**Theorem 2.2** ([8]). There is a procedure for the calculation of $G_\gamma(x, y)$, $x, y \in \Omega \setminus \bar{D}$ by means of $\{< (\Lambda_{\gamma} - \Lambda_1) f, f > | \text{supp} f \subset \Gamma\}$.

This is also an application of the Runge approximation property. Consider $D$ itself an electric conductive body having the conductivity $1 + h$. Denote the corresponding Dirichlet-to-Neumann map by $\Lambda_{D,1+h}$.

**Theorem 2.3** ([8]). There is a procedure for the calculation of $\Lambda_{D,1+h}$ by means of $G(x, y), x, y, \in \Omega \setminus \bar{D}$.

A combination of Theorem 1.1, Theorem 2.2 and Theorem 2.3 gives

**Theorem 2.4** ([8]). Assume that $h \in W^{2,\infty}(D)$. Then there is a reconstruction procedure of $h$ by means of $\{< (\Lambda_{\gamma} - \Lambda_1) f, f > | \text{supp} f \subset \Gamma\}$.

**Remark.** In the procedure we make use of only the energy $< \Lambda_{\gamma} f, f >$ for $f$ with supp $f \subset \Gamma$. 

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3 The Second Algorithm (New Application of Exponentially Growing Solution)

In contrast to Section 2, we don’t assume that \( \mathcal{D} \subset \Omega \) and that \( \Omega \setminus \mathcal{D} \) is connected. Instead of them, we assume that \( D \) is open, \( D \subset \Omega \) and that \( \partial D \) is \( C^2 \).

To describe the jump condition on \( h \) we recall

**Support Function.** The function

\[
h_D(\omega) = \sup_{x \in D} x \cdot \omega, \quad \omega \in S^2
\]

is called the support function of \( D \).

We assume that

\[
\forall \omega \in S^2 \exists C > 0 \exists \delta > 0 \quad h(x) \geq C \text{ in } D_\omega(\delta)
\]

or

\[
-h(x) \geq C \text{ in } D_\omega(\delta)
\]

where

\[
D_\omega(\delta) = \{ x \in D \mid h_D(\omega) - \delta < x \cdot \omega \leq h_D(\omega) \}.
\]

The second algorithm described below tells us how to reconstruct the support function of \( D \).

**Dirichlet Data.** Give \( \omega \in S^2 \) and take \( \omega^\perp \in S^2 \) satisfying

\[
\omega \cdot \omega^\perp = 0.
\]

Fix \( t_0 \in \mathbb{R} \). For each \( \tau > 0 \) set

\[
f_\omega(x; \tau) = \exp \{ x \cdot (\omega + i\omega^\perp) - t_0 \}, \quad x \in \partial \Omega.
\]

This is the trace of a complex valued harmonic function. Calderón used a pair of this type of harmonic functions and extracted the Fourier transform of \( \gamma - 1 \) from the Fréchet derivative \( d\Lambda_1(\gamma - 1) \). Of course, this is not an exact reconstruction procedure.

From Dirichlet data we define

**Observation Data.** Set

\[
I_\omega(\tau, t) = \langle (\Lambda_\gamma - \Lambda_1)f_\omega(\cdot; \tau), \overline{f_\omega(\cdot; \tau)} \rangle > \exp 2\tau(t_0 - t), \quad t \in \mathbb{R}
\]

and

\[
\mathcal{R}(\omega) = \{ t \mid \lim_{\tau \to \infty} I_\omega(\tau, t) = 0 \}.
\]

Then the result is

**Theorem 3.1** ([9]). For each \( \omega \in S^2 \)

\[
\mathcal{R}(\omega) = \inf \mathcal{R}(\omega) = h_D(\omega)
\]

and thus the formula

\[
\inf \mathcal{R}(\omega) = h_D(\omega)
\]
is valid.

Remark. It is possible to replace $\tau$ with $n = 1, 2, \cdots$.

The proof is reduced to studying the asymptotic behaviour of the integral

$$\int_D \exp\{2\tau x \cdot \omega\} dx$$

as $\tau \to \infty$.

To our surprise, the style of the statement of Theorem 3.1 is quite similar to a classical result in the Lax-Philips scattering theory. In [12] Lax-Philips studied the scattering theory for the wave equation

$$u_{tt} = \Delta u$$

in the complement of an obstacle $\mathcal{O}$. They proved a relationship between the scattering kernel $S(s, \theta, \omega), s \in \mathbb{R}, \theta, \omega \in S^2$ and the support function $h_{\mathcal{O}}$ of the obstacle under Dirichlet boundary condition $u = 0$ on $\partial \mathcal{O}$ or Neumann boundary condition $\partial u / \partial \nu = 0$ on $\partial \mathcal{O}$, where $\nu$ is the unit outward normal vector field to $\partial \mathcal{O}$. The one of their results is as follows. Consider the right endpoint of the supports of $S(s, \omega, -\omega)$:

$$R(\omega, -\omega) = \min \{ r | S(s, \omega, -\omega) = 0 \text{ for } s > r \}.$$

Theorem 3.2([12]). For Dirichlet or Neumann boundary conditions,

$$R(\omega, -\omega) = 2h_{\mathcal{O}}(\omega).$$

The factor 2 indicates the essential difference of the wave equation from the equation $\nabla \cdot \gamma \nabla u = 0$. However, the proof of Theorem 3.2 is also reduced to studying the asymptotic behaviour of the same integral as that of Theorem 3.1. We think that nobody expected the existence of such type of formula for the equation $\nabla \cdot \gamma \nabla u = 0$.

4 Application to an Inverse Problem for the Helmholtz Equation

As an application of the idea of the second algorithm, we consider an inverse boundary value problem for the Helmholtz equation, which is closely related to the inverse scattering problem at a fixed frequency for a sound-hard obstacle.

Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with $C^2$-boundary. Consider an open subset $D$ of $\mathbb{R}^3$ with $C^2$-boundary. We denote by $\nu$ the unit outward normal to $\partial D$ and denote by $d\sigma$ the surface measure on $\partial D$. We assume that $\mathcal{D} \subset \Omega$ and that $\Omega \setminus \mathcal{D}$ is connected. Notice that $\mathcal{D}$ is not necessarily connected. Let $k > 0$. In this section, we always consider $k$ satisfying

**Admissibility.** We say that $k$ is admissible if $0$ is not a Dirichlet eigenvalue of $\Delta + k^2$ in $\Omega$ and not an eigenvalue of the mixed problem:

$$(\Delta + k^2)u = 0 \text{ in } \Omega \setminus \mathcal{D},$$

$$u = 0 \text{ on } \partial \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$
By virtue of this assumption, we know that for any $f \in H^{1/2}(\partial \Omega)$, one can find the unique weak solution $u \in H^1(\Omega \setminus \overline{D})$ of the mixed problem:

\[(\triangle + k^2)u = 0 \text{ in } \Omega \setminus \overline{D},\]
\[u = f \text{ on } \partial \Omega,\]
\[\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.\]  

(2)

And also there exists a unique weak solution $v \in H^1(\Omega)$ of the Dirichlet problem:

\[(\triangle + k^2)v = 0 \text{ in } \Omega,\]
\[v = f \text{ on } \partial \Omega.\]  

(3)

We define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial \Omega) \longrightarrow H^{-1/2}(\partial \Omega)$ by the formula

\[< \Lambda_D f, g > = \int_{\Omega \setminus \overline{D}} \nabla u \cdot \nabla \varphi \, dx - k^2 \int_{\Omega \setminus \overline{D}} u \varphi \, dx,\]

where $g \in H^{1/2}(\partial \Omega)$, $\varphi$ is any $H^1(\Omega \setminus \overline{D})$ function satisfying $\varphi = g$ on $\partial \Omega$. And also we define the Dirichlet-to-Neumann map $\Lambda_0 : H^{1/2}(\partial \Omega) \longrightarrow H^{-1/2}(\partial \Omega)$ by the formula

\[< \Lambda_0 f, g > = \int_{\Omega} \nabla v \cdot \nabla \eta \, dx - k^2 \int_{\Omega} v \eta \, dx,\]

where $g \in H^{1/2}(\partial \Omega)$, $\eta$ is any $H^1(\Omega)$ function satisfying $\eta = g$ on $\partial \Omega$.

Now the problem considered here is

**Inverse Problem B.** Assume that $k$ and $\Omega$ are known and that $k$ is fixed. Find a reconstruction procedure of $D$ from the measurements of the "energy" $< \Lambda_D f, f >$ induced by infinitely many suitable Dirichlet data $f$.

This problem has a relationship with the inverse scattering problem at a fixed frequency. More precisely consider the reflected wave $w = w(x; d, k)$ by the sound-hard obstacle $D$ of the incident plane wave $e^{ikx \cdot d}$, $d \in S^2$. $w$ is the unique solution of

\[\triangle w + k^2 w = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D},\]
\[\frac{\partial}{\partial \nu} w = -\frac{\partial}{\partial \nu} e^{ikx \cdot d} \text{ on } \partial D,\]
\[\lim_{r \to \infty} r \left( \frac{\partial w}{\partial r} - ikw \right) = 0, r = |x|.\]

For each $\theta \in S^2$ $w$ has the asymptotic expansion

\[w(r\theta, d; k) = \frac{e^{ikx \cdot d}}{r} A(\theta, d; k) + O \left( \frac{1}{r^2} \right)\]

where $r \to \infty$. $A(\theta, d; k)$ is called the scattering amplitude. We assume that for some $R > 0$, $\Omega = B(0; R)$ and $\overline{D} \subset B(0; R/2)$.  

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We found

**Theorem 4.1** ([7], [5]). Assume that \( k \) is admissible. There is a procedure for the calculation of \( D \) from the set \( \{ A(\theta, d; k) \mid \theta, d \in S^2 \} \).

The proof is divided into two parts. The first is to show that one can calculate \( \Lambda D \) from \( A(\theta, d; k) \) given at all \( \theta, d \in S^2 \). The second is to give a procedure for the calculation of \( D \) from \( \Lambda D \). This is based on the idea of the first algorithm. In particular, we know from the first part that if one has a procedure for the calculation of some information about \( D \) from \( \Lambda D \), one automatically gets a procedure for the calculation of the same information of \( D \) from \( A(\theta, d; k) \) given at all \( \theta, d \in S^2 \) and fixed \( k \).

Here we present a result related to the second part, which is based on the idea of the second algorithm. First we specify

**Dirichlet Data.** Give \( \omega \in S^2 \) and take \( \omega \perp \in S^2 \) satisfying

\[
\omega \cdot \omega \perp = 0
\]

Fix \( t_0 \in \mathbb{R} \). For each \( \tau > 0 \) set

\[
g_\omega(x; \tau) = \exp \left\{ x \cdot z(\tau) - \tau t_0 \right\}, x \in \partial \Omega
\]

where

\[
z(\tau) = \tau \omega + i\sqrt{\tau^2 + k^2} \omega \perp.
\]

Notice that \( z(\tau) \) satisfies \( z(\tau) \cdot z(\tau) = -k^2 \) and therefore

\[
(\Delta + k^2) \exp \left\{ x \cdot z(\tau) - \tau t_0 \right\} = 0.
\]

Second we describe

**Observation Data.** Set

\[
s_\omega(\tau, t) = - \langle (\Lambda_D - \Lambda_0) g_\omega(\cdot; \tau), \overline{g_\omega(\cdot; \tau)} \rangle \exp 2\tau (t_0 - t), t \in \mathbb{R}.
\]

From the observation data we define the set

\[
\mathcal{S}(\omega) = \{ t \mid \lim_{\tau \to \infty} s_\omega(\tau, t) = 0 \}.
\]

The result is

**Theorem 4.2.** Assume that the set

\[
\{ x \in \partial D \mid x \cdot \omega = h_D(\omega) \}
\]

is consisting of the only one point and the Gaussian curvature of \( \partial D \) doesn’t vanish at the point. Then

\[
\mathcal{S}(\omega) = h_D(\omega), \infty[\]

and thus the formula

\[
\inf \mathcal{S}(\omega) = h_D(\omega),
\]

is valid.

Notice that we never make use of the precise value of the Neumann data \( \Lambda_D f \).
We say that a surface is strictly convex if its Gaussian curvature is everywhere positive. As a corollary we get

**Corollary 4.1.** Assume that $\partial D$ is connected and strictly convex. Then for any $\omega \in S^2$

$$S(\omega) = \{ h_D(\omega), \infty \}.$$

More precisely, from the proof we know that one can reconstruct $h_D(\omega)$ from the countable set

$$\{ < (\Lambda_D - \Lambda_0) g_\omega (\cdot; n), g_\omega (\cdot; n) > | n = 1, \cdots \}.$$

If one considers the Dirichlet boundary condition $u = 0$ on $\partial D$ in stead of the Neumann boundary condition $\partial u / \partial \nu = 0$ on $\partial D$, one can drop the assumption on $\omega$ and the Gaussian curvature of $\partial D$. The detail is described in [9]. We believe that one can get the same result for the Neumann boundary condition.

We give the proof of Theorem 4.2 together with that of two key lemmas listed below. Consider $u, v$ satisfying (2), (3) with $f(x) = g_\omega (x; \tau) \exp \{ \tau (t_0 - t) \} = \exp \{ x \cdot z(\tau) - \tau t \}$, $x \in \partial \Omega$, respectively. Notice that $v = \exp \{ x \cdot z(\tau) - \tau t \}$ and $u$ is independent of the choice of $t_0$.

Set

$$w_t = u - v \text{ in } \Omega \setminus D.$$

$w_t$ is in $H^1(\Omega \setminus D)$ and satisfies

$$(\triangle + k^2)w_t = 0 \text{ in } \Omega \setminus D,$$

$$w_t = 0 \text{ on } \partial \Omega,$$

$$\frac{\partial w_t}{\partial \nu} = - \frac{\partial v}{\partial \nu} \text{ on } \partial D. \quad (4)$$

**Lemma 4.1 (Key inequality).**

There exists a positive constant $C(k)$ such that for any $\omega \in S^2$, $\tau \in \mathbb{R}_+$ and $t \in \mathbb{R}$

$$2\tau^2 \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx - k^2 \int_{\Omega \setminus \overline{D}} |w_t|^2 dx$$

$$\leq s_\omega (\tau, t) \leq C(k) (\tau^2 + k^2) \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx. \quad (5)$$

**Lemma 4.2.** Assume that the set

$$\{ x \in \partial D | x \cdot \omega = h_D(\omega) \}$$

is consisting of the only one point and the Gaussian curvature of $\partial D$ doesn’t vanish at the point. Then at $t = h_D(\omega)$,

$$\lim_{\tau \to \infty} \frac{\int_{\Omega \setminus \overline{D}} |w_t|^2 dx}{2\tau^2 \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx} = 0. \quad (6)$$
Proof of Lemma 4.1.

Notice that

\[ s_\omega(\tau, t) = -\langle \Lambda_D - \Lambda_0 f, \overline{f} \rangle. \]  

(7)

From the definition of the Dirichlet-to-Neumann map we get

\[ -\langle \Lambda_D - \Lambda_0 f, \overline{f} \rangle = \int_{\Omega \setminus \overline{D}} |\nabla w_t|^2 \, dx + \int_D |\nabla v|^2 \, dx \]

\[ -k^2 \int_{\Omega \setminus \overline{D}} |w_t|^2 \, dx - k^2 \int_D |v|^2 \, dx. \]  

(8)

Since \( k \) is admissible, we have

\[ |w_t|_{H^1(\Omega \setminus \overline{D})} \leq C(k) |v|_{H^1(D)}. \]  

(9)

A combination of (7), (8), (9) and the explicit form of \( v \) yields (5).

Proof of Lemma 4.2.

Since \( k \) is admissible, one can find \( p \in H^2(\Omega \setminus \overline{D}) \) such that

\[ (\triangle + k^2) p = \overline{w}_t \text{ in } \Omega \setminus \overline{D}, \]

\[ p = 0 \text{ on } \partial \Omega, \]

\[ \frac{\partial p}{\partial \nu} = 0 \text{ on } \partial D. \]  

(10)

From the Sobolev imbedding and the estimate

\[ |p|_{H^2(\Omega \setminus \overline{D})} \leq C(k) |w_t|_{L^2(\Omega \setminus \overline{D})} \]

we have

\[ |p(x) - p(y)| \leq C(k) |x - y|^{1/2} |w_t|_{L^2(\Omega \setminus \overline{D})}, \]

\[ \sup_{x \in \Omega \setminus \overline{D}} |p(x)| \leq C(k) |w_t|_{L^2(\Omega \setminus \overline{D})}. \]  

(11)

A combination of (4), (10) and Green’s formula yields

\[ \int_{\Omega \setminus \overline{D}} |w_t|^2 \, dx = -\int_{\partial D} p \frac{\partial v}{\partial \nu} \, d\sigma(x). \]  

(12)

We denote by \( x_0 \) the only one point in the set \( \{ x \in \partial D \mid x \cdot \omega = h_D(\omega) \} \). Since

\[ \int_{\partial D} \frac{\partial v}{\partial \nu} \, d\sigma(x) = -k^2 \int_D v \, dx, \]

one can rewrite (12) in the form

\[ \int_{\Omega \setminus \overline{D}} |w_t|^2 \, dx = \int_{\partial D} \{ p(x_0) - p(x) \} \frac{\partial v}{\partial \nu} \, d\sigma(x) + k^2 p(x_0) \int_D v \, dx. \]  

(13)
From a combination of (11), (13) and the explicit form of $v$ we get

$$\int_{\Omega \setminus D} |w|^2 dx \leq C(k)\left\{ \sqrt{2\tau^2 + k^2} \int_{\partial D} |x_0 - x|^{1/2} \exp \{ \tau (x \cdot \omega - t) \} d\sigma(x) \\
+ \int_D \exp \{ \tau (x \cdot \omega - t) \} dx \right\} |w|_{L^2(\Omega \setminus D)}.$$ 

and this thus yields

$$\int_{\Omega \setminus D} |w|^2 dx \leq C(k)\left( (2\tau^2 + k^2) \left( \int_{\partial D} |x_0 - x|^{1/2} \exp \{ \tau (x \cdot \omega - t) \} d\sigma(x) \right)^2 \\
+ \left( \int_D \exp \{ \tau (x \cdot \omega - t) \} dx \right)^2 \right).$$

(14)

From the Schwarz inequality we get

$$\left( \int_D \exp \{ \tau (x \cdot \omega - t) \} dx \right)^2 \leq |D| \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx \leq |D| \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx \leq |D| 2\tau^2.$$ 

(15)

Here we note that at $t = h_D(\omega)$,

$$2\tau^2 \int_D \exp \{ 2\tau (x \cdot \omega - t) \} dx \geq C > 0, \tau \gg 1.$$ 

(16)

This is proved in [9] without making use of the assumption on $\omega$. From (16) and (6) we know that for the proof of (6), it suffices to prove

$$\lim_{\tau \to \infty} \tau \int_{\partial D} |x_0 - x|^{1/2} \exp \{ \tau (x \cdot \omega - h_D(\omega)) \} d\sigma(x) = 0.$$ 

In fact, we get

$$\tau \int_{\partial D} |x_0 - x|^{1/2} \exp \{ \tau (x \cdot \omega - h_D(\omega)) \} d\sigma(x) = O(\tau^{-1/4}) \text{ as } \tau \to \infty.$$ 

(17)

This is proved as follows. Set

$$I(\tau) = \int_{\partial D} |x_0 - x|^{1/2} \exp \{ \tau (x \cdot \omega - h_D(\omega)) \} d\sigma(x).$$

From the assumption on $\omega$ we know that the Gaussian curvature of $\partial D$ has to be positive at $x_0$. Then there exist $\epsilon > 0$, a function $f \in C_0^2(\mathbb{R}^2)$ and unit vectors $\theta, \eta$ such that:

$$\partial D \cap B(x_0, \epsilon) = \{ x = x_0 + s_1\theta + s_2\eta - f(s_1, s_2)\omega \mid s_1^2 + s_2^2 + f(s_1, s_2)^2 < \epsilon^2 \};$$

$$f(0, 0) = 0, \nabla f(0, 0) = 0;$$

$$\exists K > 0 \quad K(s_1^2 + s_2^2) \leq f(s_1, s_2) \leq K^{-1}(s_1^2 + s_2^2), s_1^2 + s_2^2 < \epsilon^2;$$

$$\theta \cdot \eta = \eta \cdot \omega = \omega \cdot \theta = 0.$$
From the compactness of \( \partial D \) we can find \( \delta > 0 \) such that
\[
\{ x \in \partial D \mid h_D(\omega) - \delta < x \cdot \omega \leq h_D(\omega) \} \subset \partial D \cap B(x_0, \epsilon).
\]
We divide the integrand of \( I(\tau) \) into the two parts:
\[
I(\tau) = \int_{x \in \partial D, x \cdot \omega \leq h_D(\omega) - \delta} |x_0 - x|^{1/2} \exp \left\{ \tau(x \cdot \omega - h_D(\omega)) \right\} d\sigma(x)
+ \int_{x \in \partial D, h_D(\omega) - \delta < x \cdot \omega \leq h_D(\omega)} |x_0 - x|^{1/2} \exp \left\{ \tau(x \cdot \omega - h_D(\omega)) \right\} d\sigma(x)
\equiv I_1(\tau) + I_2(\tau).
\]
First clearly we get
\[
I_1(\tau) \leq C \exp \{ -\tau \delta \}. \tag{18}
\]
For simplicity of description set \( s = (s_1, s_2) \). Second we get
\[
I_2(\tau) \leq \int_{|s| < \epsilon} (|s|^2 + f(s)^2)^{1/4} \exp \{ -\tau f(s) \} \sqrt{1 + \|\nabla f(s)\|^2} ds
\leq C \int_{|s| < \epsilon} |s|^{1/2} \exp \{ -\tau K|s|^2 \} ds
\leq \frac{C}{\tau^{1+1/4}} \int_{y_1^2 + y_2^2 < \tau \epsilon^2} (y_1^2 + y_2^2)^{1/4} \exp \{ -K(y_1^2 + y_2^2) \} dy_1 dy_2
\leq \frac{C}{\tau^{1+1/4}} \int_{\mathbb{R}^2} |y|^{1/2} \exp \{ -K|y|^2 \} dy = O(\tau^{-(1+\frac{1}{4})}). \tag{19}
\]
From (18) and (19) we get (17).

**Proof of Theorem 4.2.**

From (15) and (9) we get \( |h_D(\omega)|, \infty \subset \mathcal{S}(\omega) \). And from (5), (6) and (16) we know that \( h_D(\omega) \) is not in \( \mathcal{S}(\omega) \). Notice that
\[
s_\omega(\tau, t) = s_\omega(\tau, t') \exp \{ 2\tau(t' - t) \}, t, t' \in \mathbb{R} \tag{20}
\]
and in particular,
\[
s_\omega(\tau, h_D(\omega)) = s_\omega(\tau, t) \exp \{ -2\tau(t - h_D(\omega)) \}.
\]
Therefore any \( t < h_D(\omega) \) never belong to \( \mathcal{S}(\omega) \). In fact, for \( t < h_D(\omega) \) we get
\[
\exists C > 0 \ s_\omega(\tau, t) \geq C \exp \{ 2\tau(h_D(\omega) - t) \}, \tau >> 1.
\]
This thus yields \( \mathcal{S}(\omega) \subset [h_D(\omega), \infty[. \)
5 Remark

In section 4 we excluded the case when $k = 0$. This case is closely related to the electric impedance tomography. It corresponds to considering an electrostatic problem for a conductor consisting of finitely many inhomogeneities of zero conductivity, embedded in a spatial reference medium. However, the treatment of this case is simpler and easier. We can get a complete answer without making use of any assumption on $\omega \in S^2$. And also our method covers a similar problem for the equation $\nabla \cdot \gamma \nabla u = 0$ with a variable coefficient $\gamma$. Here we briefly describe the result and outline of the proof.

Let $\Omega$ be a bounded domain of $\mathbb{R}^m (m = 2, 3)$ with Lipschitz boundary. We consider $\Omega$ as an inhomogeneous isotropic conductive medium containing an object $D$, whose boundary is perfectly insulated. We assume that the conductivity $\gamma$ of $\Omega$ satisfies

$$\gamma - 1 \in C^0_0(\mathbb{R}^m), \; \gamma > 0.$$  

And also assume that $D$ is open, $\overline{D} \subset \Omega$ and that $\partial D$ is Lipschitz, $C^2$ if $m = 2, 3$, respectively. We assume that $\gamma$ is known and $D$ is unknown. The problem is to reconstruct $D$ from the measurements of electric currents $\gamma \partial u/\partial \nu$ at the boundary, $\partial \Omega$ induced by the voltage potentials $u$ at $\partial \Omega$. This process is described by means of the Dirichlet-to-Neumann map $\Lambda_D$ which takes Dirichlet data into Neumann data:

$$\Lambda_D f = \gamma \frac{\partial u}{\partial \nu} |_{\partial \Omega}$$  

with

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega \setminus \overline{D},$$

$$u = f \text{ on } \partial \Omega,$$

$$\gamma \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D.$$  

(21)

We set $\Lambda_0 = \Lambda_D$ for $D = \emptyset$.

We specify

**Dirichlet data.** Give $\omega \in S^{m-1}$ and take $\omega^\perp \in S^{m-1}$ satisfying

$$\omega \cdot \omega^\perp = 0.$$  

Set

$$z = \omega + i \omega^\perp.$$  

From [17], [18] we know that if $\tau > C = C(\gamma) \gg 1$, there exists a solution of $\nabla \cdot \gamma \nabla v = 0$ in $\Omega$ having the form

$$v(x) = \frac{1}{\sqrt{\gamma(x)}} \exp \{\tau(x \cdot z)\} (1 + \Psi(x; \tau z)).$$

$\Psi$ is in $H^4(\Omega)$ and satisfies

$$|\Psi|_{L^\infty(\Omega)} = |\nabla \Psi|_{L^\infty(\Omega)} = O \left(\frac{1}{\tau}\right) \text{ as } \tau \to \infty.$$  

(22)
Fix \( t_0 \in \mathbb{R}^m \) and for \( \tau > C(\gamma) \) set
\[
f_\omega(x; \tau) = \frac{1}{\sqrt{\gamma(x)}} \exp \{ \tau(x \cdot z - t_0) \} (1 + \Psi(x; \tau z)), x \in \partial\Omega.
\]
Notice that one can calculate \( f_\omega(\cdot; \tau) \) from \( \gamma \) and \( \Omega \). Then we define
\[
\text{Observation data.}
\]
\[
s_\omega(\tau, t) = -\langle (\Lambda_D - \Lambda_0)f_\omega(\cdot; \tau), \overline{f_\omega(\cdot; \tau)} \rangle > \exp 2\tau(t_0 - t), t \in \mathbb{R}, \tau > C(\gamma)
\]
and
\[
S(\omega) = \{ t \mid \lim_{\tau \to \infty} s_\omega(\tau, t) = 0 \}.
\]
The result is
\[
\text{Theorem 5.1. For any} \, \omega \in S^{m-1}
\]
\[
S(\omega) = ]h_D(\omega), \infty[.
\]
The proof is an easy consequence of following three facts and the technique in [9]:
\[
- < (\Lambda_D - \Lambda_0)f, \overline{f} > = \int_{\Omega \setminus D} \gamma |\nabla(u - v)|^2 dx + \int_D \gamma |\nabla v|^2 dx
\]
where \( u \) solves (21) and \( v \) solves
\[
\nabla \cdot \gamma \nabla v = 0 \text{ in } \Omega
\]
\[
v = f \text{ on } \partial \Omega;
\]
\[
\tau^2 \int_D \exp \{2\tau(x \cdot \omega - h_D(\omega))\} dx \geq C > 0, \tau > C >> 1;
\]
\[
\int_D |\nabla v|^2 dx \geq C^{-1} \tau^2 \int_D \exp \{2\tau(x \cdot \omega - h_D(\omega))\} dx, \tau > C >> C(\gamma)
\]
where
\[
v(x) = \frac{1}{\sqrt{\gamma(x)}} \exp \{ \tau(x \cdot z - h_D(\omega)) \} (1 + \Psi(x; \tau z)).
\]
(23) comes from the regularity of \( \partial D([9]) \). (24) comes from (22).
In fact, from these facts we know that one can reconstruct \( h_D(\omega) \) from the countable set
\[
\{ < (\Lambda_D - \Lambda_0)f_\omega(\cdot; n), \overline{f_\omega(\cdot; n)} > \mid n = n_0, n_0 + 1, \ldots, n_0 > C(\gamma) \}.
\]
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