Inverse scattering problems
and
the enclosure method

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Abstract
We show that the enclosure method introduced here is applicable to inverse
scattering problems in two dimensions. Using the single set of the Cauchy data of a
solution of the Helmholtz equation, we give some formulae for extracting the value
of the support functions of unknown sound-hard polygonal obstacles and piecewise
linear cracks from the set.
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1 Introduction
The aim of this paper is to show that the enclosure method introduced by the author is
applicable to inverse scattering problems in two dimensions.
First we review the enclosure method in a typical inverse problem (see [11] for a survey
on the method). Let $B_R(0)$ denote the open disc centered at 0 with radius $R$. Let $D$
be an open set of $\mathbb{R}^2$ with $\overline{D} \subset B_R(0)$. We consider only the case when $B_R(0) \setminus \overline{D}$ is
connected.

Definition 1.1. We say that $D$ is polygonal if $D = D_1 \cup D_2 \cup \cdots \cup D_m$; each $D_j$ is a
simply connected open set and polygon; $\overline{D}_j \cap \overline{D}_{j'} = \emptyset$ for $j \neq j'$.
Assume that $D$ is polygonal. Let $u \in H^1(B_R(0) \setminus \overline{D})$ be nonconstant and satisfy the
Laplace equation $\Delta u = 0$ in $B_R(0) \setminus \overline{D}$ with boundary condition $\partial u / \partial \nu = 0$ on $\partial D$.
Here $\nu$ is the outward normal relative to $B_R(0) \setminus \overline{D}$. In [9] we considered the problem
of extracting information about the location and shape of $D$ from the single set of the
Cauchy data $u$ and $\partial u / \partial \nu$ on $|x| = R$. Define
$$h_D(\omega) = \sup_{x \in D} x \cdot \omega, \; \omega \in S^1.$$ $h_D$ is called the support function of $D$. We say that the direction $\omega \in S^1$ is regular with
respect to $D$ if the set $\{ x \mid x \cdot \omega = h_D(\omega) \} \cap \partial D$ consists of only one point. The following
theorem tells us how to extract the value of the support function of unknown $D$ from the
Cauchy data of $u$ on $|x| = R$. 

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Theorem 1.1([9]). Let $\omega$ be regular with respect to $D$. Assume that
\[
\text{diam } D < \text{dis}(D, \partial B_R(0)).
\] (1.1)

Then the formula
\[
\lim_{\tau \to \infty} \log \left| \int_{|x|=R} \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right| d\sigma(x) / \tau = h_D(\omega),
\] (1.2)

is valid where $v = e^{\tau x \cdot (\omega + i \omega^\perp)}$ and $\omega^\perp = (\omega_2, -\omega_1)$ for $\omega = (\omega_1, \omega_2)$.

Note that there is no restriction of $u$ or $\partial u/\partial \nu$ on $|x| = R$ at the cost of the condition (1.1); $v$ satisfies the Laplace equation $\triangle v = 0$. In [13] a numerical implementation of an algorithm based on the formula (1.2) has been done.

In this paper, we establish a formula similar to (1.2) in the case when $u$ satisfies the Helmholtz equation $\triangle u + k^2 u = 0$ in $\mathbb{R}^2 \setminus D$, the boundary condition $\partial u / \partial \nu = 0$ on $\partial D$ and a condition at infinity that expresses $u \sim e^{ikx \cdot d}$ as $|x| \to \infty$ in a suitable sense. Note that $D$ is a model of the section of the sound-hard obstacles in three-dimensions. Both $d \in S^1$ and $k > 0$ are fixed. This means that our method makes use of only one incident plane wave at fixed wave number. For another approach sharing this feature see [14]. Therein a summary of several methods in inverse obstacle scattering problems is also given.

The enclosure method presented in this paper makes use of a special solution of the Helmholtz equation with a large parameter $\tau$ that has a different behaviour across a line as $\tau \to \infty$. The property affects the different behaviour of the indicator function (see (2.3)) across the line $x \cdot \omega = h_D(\omega)$. The main task is to study the asymptotic behavior of the integral as $\tau \to \infty$:
\[
e^{-\tau h_D(\omega)} \int_{|x|=R} \left( \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma(x)
\] (1.3)

where
\[
v = e^{\tau x \cdot (\omega + i \sqrt{\tau^2+k^2} \omega^\perp)}.
\] (1.4)

Note that $v$ satisfies the Helmholtz equation $\triangle v + k^2 v = 0$.

In [12] we considered a similar problem in which $D$ is replaced with sound-hard linear cracks and studied an integral with large parameter $\tau$ similar to (1.3). It turned out that it is quite difficult to determine the complete asymptotic expansion as $\tau \to \infty$.

In this paper, we introduce the new parameter $s$ given by
\[
s = \sqrt{\tau^2 + k^2} + \tau.
\] (1.5)

We see that $\tau$ is represented by $s$ rationally as
\[
\tau = \frac{s}{2} \left\{ 1 - \left( \frac{k}{s} \right)^2 \right\}.
\] (1.6)

Then we discovered that (1.3) has a simple complete asymptotic expansion as $s \to \infty$. And using the argument similar to that of [9] and the special feature of $u$, one concludes that there is a nonzero coefficient in the expansion of (1.3) as $s \to \infty$. Then
we immediately obtain a formula similar to (1.2). We do not need to assume (1.1). Moreover, we give the corresponding formulae in the case when \( D \) is replaced with the sound-hard piecewise linear cracks. However, we have to impose a restriction on incident direction \( d \).

It should be pointed out that in the case when \( k = 0 \) using the integral

\[
\int_{|x|=R} \left( \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma(x)
\]

for special harmonic functions \( v \) and a solution \( u \) of the governing equation, Andrieux-Abda [1] gave a reconstruction formula of unknown cracks on a single unknown line (in two-dimensions), plane (in three-dimensions) from the Cauchy data of \( u \) on \( |x| = R \) provided the integral of the jump of \( u \) on the cracks does not vanish. Those harmonic functions are given by harmonic polynomials of degree one, two and harmonic functions of separation of variables. Their method of use of integral (1.7) is divided into two parts: extracting algebraic equations that determine the equation of the line where the unknown cracks are on: extracting the jump of \( u \) on the line reconstructed in the first part.

We do not know how to apply their method to the case when \( k > 0 \) or the cracks do not exist on the single line. They call (1.7) the reciprocity gap functional.

Remark 1.1. We give some explanation about new parameter \( s \). From (1.5) and (1.6), one knows that \( \tau \omega + i\sqrt{\tau^2 + k^2} \omega^\perp \) in (1.4) becomes

\[
\xi(s) \equiv \frac{s}{2} \left\{ 1 - \left( \frac{k}{s} \right)^2 \right\} \omega + \frac{i}{2} \left\{ 1 + \left( \frac{k}{s} \right)^2 \right\} \omega^\perp.
\]

Extend \( \xi(s) \) for an arbitrary nonzero complex number \( z \) just by replacing \( s \) with \( z \). It is easy to see that the map \( \xi(\cdot) : C \setminus \{0\} \rightarrow \mathbb{C}^2 \) is injective and its image coincides with the set

\[
\{ a\omega + ib\omega^\perp \mid a, b \in C, \ (a\omega + ib\omega^\perp) \cdot (a\omega + ib\omega^\perp) = -k^2 \}.
\]

This fact has been described and used in [5] for a different purpose. Note that (1.5) is equivalent to the equation \( \xi(s) = \tau \omega + i\sqrt{\tau^2 + k^2} \omega^\perp \).

It would be interested to apply the method presented in this paper to the corresponding problems: for the equation \( \nabla \cdot \gamma \nabla u + k^2 u = 0 \) where \( \gamma \) is piecewise constant (see [10] for the case when \( k = 0 \) which gives us a quantitative explanation of Friedman-Isakov’s uniqueness theorem ([6])); for the Maxwell equations or in three-dimensions. Those belong to our future works. Anyway it is sure that the enclosure method provides us with many mathematical problems.

The outline of this paper is as follows. In Section 2 we describe the definition of the indicator function and the two main results. These are proved in Section 5. The proof is based on the complete asymptotic expansion formula of the indicator function as derived in Section 4. To establish the expansion formula we require an expansion formula of an oscillatory integral with a large parameter modulo rapidly decreasing with respect to the parameter; this formula is found in Section 3.

2 Description of the two main results

We divide the results into two cases. Let \( d \in S^1 \) and \( k > 0 \).
**Case I. The sound-hard polygonal obstacles.** In this subsection we assume that $D$ is an open set of $\mathbb{R}^2$ with $D \subset B_R(0)$ and polygonal; $B_R(0) \setminus D$ is connected. Let $u \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ satisfy:

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D};$$

$$u \in H^1(B_R(0) \setminus \overline{D}) \text{ and satisfies, for all } \varphi \in H^1(B_R(0) \setminus \overline{D}) \text{ with } \varphi = 0 \text{ on } |x| = R$$

$$\int_{B_R(0) \setminus \overline{D}} (\nabla u \cdot \nabla \varphi - k^2 u \varphi) dx = 0; \tag{2.1}$$

the condition at infinity

$$|\nabla(u - e^{ikx \cdot d})| \rightarrow 0 \text{ as } |x| \rightarrow \infty. \tag{2.2}$$

Note that $u$ is identified with a smooth function in $\mathbb{R}^2 \setminus \overline{D}$ and thus (2.2) has a meaning. (2.1) is a weak formulation of the boundary condition $\partial u / \partial \nu = 0$ on $\partial D$. Note that $\nu$ stands for the unit outward normal relative to $B_R(0) \setminus \overline{D}$. We do not assume anything more about $u$. This is the minimum requirement for the enclosure method.

**Remark 2.1.** One can easily show that the radiation condition for the reflected wave $w = u - e^{ikx \cdot d}$

$$\sqrt{r}(\frac{\partial w}{\partial r} - ikw) \rightarrow 0 \text{ as } r = |x| \rightarrow \infty,$$

implies (2.2) keeping other conditions on $u$. The proof of this fact proceeds along the same lines as Appendix of [12]. See also [4] for the radiation condition and its consequences; applying the variational method presented in [8] to our simpler situation, one can prove the existence of a solution of the Helmholtz equation $u \in L^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})$ satisfying the radiation condition (and thus (2.2)) and (2.1). The same comments work also for Case II.

**Definition 2.1.** Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. Define the indicator function $I_\omega(\tau, t)$ by the formula

$$I_\omega(\tau, t) = e^{-\tau t} \int_{|x|=R} \left( \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma(x) \tag{2.3}$$

where $-\infty < t < \infty$, $\tau > 0$ and $v$ is given by (1.4).

The first result is

**Theorem 2.1.** Let $\omega$ be regular with respect to $D$. Then the formula

$$\lim_{\tau \rightarrow -\infty} \frac{\log |I_\omega(\tau, 0)|}{\tau} = h_D(\omega), \tag{2.4}$$

is valid. Moreover, we have

- if $t \geq h_D(\omega)$, then $\lim_{\tau \rightarrow -\infty} |I_\omega(\tau, t)| = 0$;
- if $t < h_D(\omega)$, then $\lim_{\tau \rightarrow -\infty} |I_\omega(\tau, t)| = \infty$.

Needless to say, the counting number of $\omega$ that is not regular with respect to $D$ is finite. Therefore we do not need to worry about the choice of such $\omega$. There is no additional assumption on $D$ like (1.1) and the incident direction $d$. 

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Case II. The sound-hard piecewise linear cracks. Let \( \Sigma \) be the union of finitely many disjoint closed piecewise linear segments \( \Sigma_1, \Sigma_2, \ldots, \Sigma_m \). Assume that there exists a simply connected open set \( D \) such that \( D \) is a polygon and each \( \Sigma_j \) consists of sides of \( D \). We assume that \( \overline{D} \subset B_R(0) \). We denote by \( \nu \) the unit outward normal relative to \( B_R(0) \setminus \overline{D} \). Denote by \( H^1(B_R(0) \setminus \Sigma) \) the set of all \( L^2(B_R(0)) \) functions \( u \) such that, if \( x \) satisfies (2.3) for sufficiently small \( \eta \), then, for \( \tau \rightarrow \infty \), one has:

\[
|I_\omega(\tau, t)| = 0.
\]

If there is an end point \( x_0 \) of some \( \Sigma_j \) such that \( x_0 \cdot \omega = h_\Sigma(\omega) \), then, for \( d \) that is not perpendicular to \( \nu \) on \( \Sigma_j \) near the point, the same conclusions as above are valid.

Note that \( \nu \) on \( \Sigma_j \cap B_\eta(x_0) \) for sufficiently small \( \eta > 0 \) becomes a constant vector if \( x_0 \) is an end point of \( \Sigma_j \).

Theorem 2.2 is an extension of a result in [12]. Therein we did not introduce \( s \) given by (1.5). This yields an unnecessary complexity of the analysis of the asymptotic behavior of the indicator function at \( t = h_\Sigma(\omega) \) when the line \( x \cdot \omega = h_\Sigma(\omega) \) hits a vertex of \( D \) that belongs to some \( \Sigma_j \) and not an end point of \( \Sigma_j \). Thus we could treat only the case when all \( \Sigma_j \) are given by closed segments.

3 An expansion formula of an oscillatory integral

Let \(-\pi < \theta < 0, \mu \geq 0 \) and \( \eta > 0 \). Set

\[
I_\mu(\tau, \theta) = \int_0^\eta J_\mu(kr) e^{ir \sin \theta} e^{i \sqrt{r^2 + k^2} \cos \theta} dr
\]

where \( \tau > 0 \) and \( J_\mu \) denotes the Bessel function of order \( \mu \) given by the formula

\[
J_\mu(z) = (\frac{z}{2})^\mu \frac{\Gamma(\mu + n + 1)}{\Gamma(n + 1 + \mu)} (\frac{z}{2})^{2n}.
\]

In this section we prove the following.
Proposition 3.1. Let $\tau$ be given by (1.6). As $s \to \infty$ the formula
\[
(\tau \cos \theta - i\sqrt{\tau^2 + k^2} \sin \theta) I_\mu(\tau, \theta) = \frac{ie^{i(\theta + \frac{\pi}{2})} l^\mu}{s^\mu} + O(s^{-\infty}),
\] (3.2)
is valid.

This section is divided into three parts:
(1) description of three lemmas;
(2) a proof of Proposition 3.1 by using those three lemmas;
(3) proofs of three lemmas.

The followings are needed for the proof of Proposition 3.1.

Lemma 3.1. For each $l' = 0, 1, \ldots$
\[
I_\mu(\tau, \theta)
\]
\[
= \sum_{j=0}^{l'} \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \left(\frac{k}{2}\right)^{2j+\mu} \int_0^\infty r^{2j+\mu} e^{\tau r \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr + O(\frac{1}{\tau^{1+2(l'+1)+\mu}}),
\] (3.3)

Lemma 3.2. Let $-\pi < \theta < 0$ and $\sigma \geq 0$. For each $l = 0, 1, \ldots$ as $s \to \infty$ the formula
\[
\int_0^\infty r^\sigma e^{\tau r \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr = \sum_{n=0}^l \frac{L_{\sigma,n}(\theta)}{s^{\sigma + 2n + 1}} + O\left(\frac{1}{s^{\sigma + 2(l+1)+1}}\right),
\] (3.4)
is valid where
\[
L_{\sigma,n}(\theta) = i e^{i\theta} e^{i(\theta + \frac{\pi}{2})} \sigma^{\sigma + 1} (-k^2 e^{2i\theta})^n \frac{\Gamma(\sigma + n + 1)}{n!}.
\]

Lemma 3.3.
\[
\sum_{n_1+n_2=n} \frac{(-1)^{n_1} \Gamma(n + 1 + n_2 + \mu)}{n_1! n_2! \Gamma(1 + n_2 + \mu)} = (-1)^n.
\] (3.5)

Proof of Proposition 3.1. A combination of (3.3) and (3.4) for $\sigma = 2j + \mu$ gives
\[
I_\mu(\tau, \theta)
\]
\[
= \sum_{j=0}^{l'} \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \left(\frac{k}{2}\right)^{2j+\mu} \left\{ \sum_{n=0}^l \frac{L_{2j+\mu,n}(\theta)}{s^{2j+\mu+2n+1}} + O\left(\frac{1}{s^{2j+\mu+2(l+1)+1}}\right) \right\} + O(\frac{1}{s^{1+2(l'+1)+\mu}})
\]
\[
= \sum_{j=0}^{l'} \sum_{n=0}^l \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \left(\frac{k}{2}\right)^{2j+\mu} \frac{L_{2j+\mu,n}(\theta)}{s^{2(n+j)+\mu+1}} + O\left(\frac{1}{s^{2(l+1)+\mu+1}}\right) + O(\frac{1}{s^{2(l'+1)+\mu+1}}).
\] (3.6)

Now let $l = l'$. Then (3.6) becomes
\[
I_\mu(\tau, \theta)
\]
\[
= \sum_{n=0}^l \sum_{n_1+n_2=n} \frac{(-1)^{n_2}}{n_2! \Gamma(1 + n_2 + \mu)} \left(\frac{k}{2}\right)^{2n_2+\mu} L_{2n_2+\mu,n_1}(\theta) \times \frac{1}{s^{2n_2+\mu+1}} + O\left(\frac{1}{s^{2(l+1)+\mu+1}}\right).
\] (3.7)
Write
\[
\sum_{n_1+n_2=n} (-1)^{n_2} \frac{\left(k^2 \right)^{2n_2+\mu}}{n_2! \Gamma(1+n_2+\mu)} L_{2n_2+\mu,n_1}(\theta)
\]
\[
= \sum_{n_1+n_2=n} (-1)^{n_2} \frac{\left(k^2 \right)^{2n_2+\mu}}{n_2! \Gamma(1+n_2+\mu)} \mu \ k^\mu
\]
\[
\times i e^{i\theta} e^{i(\theta+\frac{\pi}{2})}(2n_2+\mu)2^{2n_2+\mu+1}(-k^2 e^{2i\theta})^{n_2} \frac{\Gamma(2n_2+\mu+n_1+1)}{n_1!}.
\]

Now from (3.5), (3.7) and (3.8) one obtains
\[
I_\mu(\tau, \theta) = 2i e^{i\theta} e^{i(\theta+\frac{\pi}{2})} k^\mu (-k^2 e^{2i\theta}) \sum_{n_1+n_2=n} (-1)^{n_2} \frac{\Gamma(n+1+n_2+\mu)}{n_1!n_2! \Gamma(1+n_2+\mu)}.
\]

Write
\[
\tau \cos \theta - i \sqrt{\tau^2 + k^2} \sin \theta
\]
\[
= \tau \cos \theta - i \tau \sin \theta - i(\sqrt{\tau^2 + k^2} - \tau) \sin \theta
\]
\[
= s \left\{ 1 - \left(\frac{k}{s} \right)^2 e^{-i\theta} - \frac{k^2}{s} \sin \theta \right\}
\]
\[
= \frac{e^{-i\theta}}{2} - \frac{i k^2}{s} \sin \theta - \frac{k^2}{s} e^{-i\theta}
\]
\[
= \frac{e^{-i\theta}}{2} - \frac{k^2}{s} \left(2i \sin \theta + e^{-i\theta} \right)
\]
\[
= \frac{1}{2} \left\{ s e^{-i\theta} - \frac{1}{s} k^2 e^{i\theta} \right\}.
\]

Then from (3.9) we obtain
\[
(\tau \cos \theta - i \sqrt{\tau^2 + k^2} \sin \theta) I_\mu(\tau, \theta)
\]
\[
= \frac{1}{2} \left\{ s e^{-i\theta} - \frac{1}{s} k^2 e^{i\theta} \right\} \left\{ 2i e^{i\theta} e^{i(\theta+\frac{\pi}{2})} k^\mu \sum_{n=0}^{l} (-k^2 e^{2i\theta})^{n} \sum_{n=0}^{l} \frac{(k^2 e^{2i\theta})^{n}}{s^{2n+\mu+1}} + O(\frac{1}{s^{2(l+1)+\mu+1}}) \right\}
\]
\[
= i e^{i(\theta+\frac{\pi}{2})} k^\mu \sum_{n=0}^{l} \frac{(k^2 e^{2i\theta})^{n}}{s^{2n+\mu}} + O(\frac{1}{s^{2(l+1)+\mu+1}}) - i k^2 e^{2i\theta} e^{i(\theta+\frac{\pi}{2})} k^\mu \sum_{n=0}^{l} \frac{(k^2 e^{2i\theta})^{n+1}}{s^{2(n+1)+\mu+2}} + O(\frac{1}{s^{2(l+1)+\mu+2}})
\]
\[
= i e^{i(\theta+\frac{\pi}{2})} k^\mu \left( \sum_{n=0}^{l} \frac{(k^2 e^{2i\theta})^{n}}{s^{2n+\mu}} - \sum_{n=0}^{l-1} \frac{(k^2 e^{2i\theta})^{n+1}}{s^{2(n+1)+\mu+2}} + O(\frac{1}{s^{2(l+1)+\mu}}) \right)
\]
\[
= i e^{i(\theta+\frac{\pi}{2})} k^\mu \frac{1}{s^\mu} + O(\frac{1}{s^{2(l+1)+\mu}}).
\]
Since \( l \) is arbitrary, we obtain the desired formula (3.2). This completes the proof of Proposition 3.1.

\[
\square
\]

Proof of Lemma 3.1. Write
\[
J_\mu(x) = \left( \frac{x}{2} \right)^\mu \sum_{j=0}^\nu (-1)^j j! \Gamma(1 + j + \mu) \left( \frac{x}{2} \right)^{2j} + R'_\nu(x).
\]

Then we have
\[
I_\mu(\tau, \theta) = \int_0^\infty J_\mu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr - \int_\eta^\infty J_\mu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr
\]
\[
= \sum_{j=0}^\nu \frac{(-1)^j}{j! \Gamma(1 + j + \mu)} \int_0^\infty \left( \frac{kr}{2} \right)^{\mu+2j} e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr
\]
\[
+ \int_0^\infty R'_\nu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr - \int_\eta^\infty J_\mu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr.
\]

By [15], p.59, Ex.9.6., we have
\[
|J_\mu(x)| \leq \left( \frac{x}{2} \right)^\mu \frac{1}{\Gamma(1 + \mu)}.
\]

Using this together with integration by parts, we have the estimation of the third term on the right-hand side of (3.10):
\[
|\int_\eta^\infty J_\mu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr| \leq \left( \frac{k}{2} \right)^\mu \frac{1}{\Gamma(1 + \mu)} \int_\eta^\infty \tau^\mu e^{r\tau \sin \theta} dr
\]
\[
= \left( \frac{k}{2} \right)^\mu \frac{1}{\Gamma(1 + \mu)} \left( \frac{1}{\tau} \right)^{(1+\mu)} \int_\eta^\infty w^\mu e^w \sin \theta dw
\]
\[
= O\left( \frac{e^{\eta \tau \sin \theta}}{\tau} \right).
\]

On the other hand, since as \( x \to 0 \)
\[
R'_\nu(x) = O\left( \left( \frac{x}{2} \right)^{2(\nu'+1)+\mu} \right),
\]
for sufficiently small \( \eta_\nu > 0 \) we have
\[
|R'_\nu(x)| \leq C(\nu') \left( \frac{x}{2} \right)^{2(\nu'+1)+\mu} 0 \leq x \leq \eta_\nu.
\]

This yields
\[
|\int_0^\infty R'_\nu(kr)e^{r\tau \sin \theta} e^{i\sqrt{\tau^2 + k^2} r \cos \theta} dr|
\]
\[
\leq C(\nu') \int_{\eta_\nu/k}^{\eta_\nu} r^{2(\nu'+1)+\mu} e^{r\tau \sin \theta} dr + \int_{\eta_\nu/k}^\infty |R'_\nu(kr)|e^{r\tau \sin \theta} dr.
\]

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Note that

\[
\int_0^{\eta/\tau} r^{2(l'+1)+\mu} e^{\tau r \sin \theta} \, dr = \left(\frac{1}{\tau}\right)^{2(l'+1)+1+\mu} \int_0^{\eta/\tau} w^{2(l'+1)+\mu} e^{w \sin \theta} \, dw \leq \left(\frac{1}{\tau}\right)^{2(l'+1)+1+\mu};
\]

\[
\int_0^{\eta/\tau} w^{2(l'+1)+\mu} e^{w \sin \theta} \, dw = C' \left(\frac{1}{\tau}\right)^{2(l'+1)+1+\mu};
\]

\[
\int_0^{\eta/\tau} |R\eta(\tau)| e^{\tau r \sin \theta} \, dr \leq \int_0^{\eta/\tau} |J\eta(\tau)| e^{\tau r \sin \theta} \, dr + \sum_{j=0}^{l'} \frac{1}{j! \Gamma(1+j+\mu)} \int_0^{\eta/\tau} \left(\frac{kr}{2}\right)^{2j+\mu} e^{\tau r \sin \theta} \, dr.
\]

Then from those and (3.13) one gets

\[
\int_0^{\eta/\tau} |R\eta(\tau)| e^{\tau r \sin \theta} \, dr = \int_0^{\eta/\tau} |J\eta(\tau)| e^{\tau r \sin \theta} \, dr + \sum_{j=0}^{l'} \frac{1}{j! \Gamma(1+j+\mu)} \int_0^{\eta/\tau} \left(\frac{kr}{2}\right)^{2j+\mu} e^{\tau r \sin \theta} \, dr = O\left(\tau^{2j+\mu} e^{\eta r \sin \theta/k}\right).
\]

Therefore from (3.10), (3.12) and (3.14) we obtain (3.3).

\[\Box\]

**Proof of Lemma 3.2.** Define

\[
C_0^\alpha = 1,
\]

\[
C_n^\alpha = \frac{\alpha(\alpha+1)\cdots \{\alpha+(n-1)\}}{n!}.
\]

Since

\[
\frac{1}{(1-x)^\alpha} = \sum_{n=0}^{\infty} C_n^\alpha x^n, |x| < 1,
\]

from (1.6) we have

\[
\frac{1}{\tau^\alpha} = 2^\alpha \sum_{n=0}^{l} C_n^\alpha \frac{k^{2n}}{s^{2n+\alpha}} + O\left(\frac{1}{s^{2(l+1)+\alpha}}\right).
\]

Since

\[
\sqrt{\tau^2 + k^2} - \tau = \frac{k^2}{s},
\]
one can write
\[ \int_0^\infty r^\sigma e^{\tau r \sin \theta} e^{i\sqrt{\tau^2+k^2}r \cos \theta} dr = \int_0^\infty r^\sigma e^{\tau r \sin \theta + i\tau r \cos \theta} e^{i(\sqrt{\tau^2+k^2}r - \tau \cos \theta)} dr. \]  
(3.16)
\[ = \int_0^\infty r^\sigma e^{\tau r \sin \theta + i\tau r \cos \theta} \left( \frac{k^2}{s} e^{-r \cos \theta} \right) dr. \]

Note that
\[ \frac{k^2}{s} e^{-r \cos \theta} = \sum_{n=0}^l i^n k^{2n} \cos^n \theta \frac{r^n + O(r^{l+1})}{n!}. \]

A combination of this with (3.16) yields
\[ \int_0^\infty r^\sigma e^{\tau r \sin \theta} e^{i\sqrt{\tau^2+k^2}r \cos \theta} dr = \sum_{n=0}^l i^n k^{2n} \cos^n \theta \frac{r^n + O(r^{l+1})}{n!} \int_0^\infty r^{\sigma + n} e^{\tau r \sin \theta + i\tau r \cos \theta} dr + O(\frac{1}{s^{l+1}} \int_0^\infty r^{\sigma + l+1} e^{\tau r \sin \theta} dr). \]  
(3.17)

Since \(-\pi < \theta < 0\), we have
\[ \int_0^\infty r^{\sigma + l+1} e^{\tau r \sin \theta} dr = \frac{1}{\tau^{\sigma + l+2}} \int_0^\infty w^{\sigma + l+1} e^{w \sin \theta} dw = O(\frac{1}{s^{l+1}}). \]

On the other hand we already know the formula (see [9])
\[ \int_0^\infty r^{\sigma + n_1} e^{\tau r \sin \theta + i\tau r \cos \theta} dr = \frac{1}{\tau^{\sigma + n_1 + 1}} \int_0^\infty w^{\sigma + n_1} e^{w \sin \theta + i w \cos \theta} dw = \frac{i e^{i\pi/2} (\gamma + n_1) e^{i\theta (\gamma + n_1)} \Gamma(1 + \gamma + n_1)}{\tau^{\sigma + n_1 + 1}}. \]

Therefore (3.17) becomes
\[ \int_0^\infty r^\sigma e^{\tau r \sin \theta} e^{i\sqrt{\tau^2+k^2}r \cos \theta} dr = \sum_{n_1=0}^l i^{n_1+1} k^{2n_1} \cos^n \theta e^{i\pi/2 (\gamma + n_1)} e^{i\theta (\gamma + n_1)} \Gamma(1 + \gamma + n_1) + O(\frac{1}{s^{\sigma + 2(l+1)+1}}). \]  
(3.18)

Define
\[ K_{\alpha,n}(\theta) = \frac{i^{n+1} k^{2n} \cos^n \theta e^{i\pi/2} e^{i\theta} e^{i\alpha} \Gamma(1 + \alpha)}{n!}. \]
A combination of (3.15) for $\alpha = \sigma + n_1 + 1$ and (3.18) gives

$$
\int_0^\infty r^\sigma e^{ir\sin \theta} e^{i\sqrt{r^2 + k^2} r \cos \theta} dr
$$

$$
= \sum_{n_1=0}^{l_1} \frac{K_{\sigma+n_1,n_1}(\theta)}{\sigma^{n_1+\sigma+n_1+1}} + O\left(\frac{1}{\sigma^{\sigma+2(l_1+1)}}\right)
$$

$$
= \sum_{n_1=0}^{l_1} \sum_{n_2=0}^{l_2} \frac{K_{\sigma+n_1,n_1}(\theta)2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}k^{2n_2}}{\sigma^{\sigma+2(n_1+n_2)+1}} + O\left(\frac{1}{\sigma^{\sigma+2(l_1+1)}}\right) + O\left(\frac{1}{\sigma^{\sigma+2(l_1+1)}}\right).
$$

(3.19)

Now let $l_1 = l_2 = l$. Then (3.19) becomes

$$
\int_0^\infty r^\sigma e^{ir\sin \theta} e^{i\sqrt{r^2 + k^2} r \cos \theta} dr
$$

$$
= \sum_{n_1+n_2=n} \frac{K_{\sigma+n_1,n_1}(\theta)2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}k^{2n_2}}{\sigma^{\sigma+2(n_1+n_2)+1}} + O\left(\frac{1}{\sigma^{\sigma+2(l+1)+1}}\right).
$$

(3.20)

One can write

$$
\sum_{n_1+n_2=n} K_{\sigma+n_1,n_1}(\theta)2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}k^{2n_2}
$$

$$
= \sum_{n_1=n_2=n} \frac{i^{n_1+n_2}k^{2n_1}\cos^{n_1}\theta e^{i\frac{\pi}{2}(\sigma+n_1)} e^{i\theta} e^{i\theta(\sigma+n_1)} \Gamma(\sigma + n_1 + 1) 2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}k^{2n_2}}{n_1!}
$$

$$
= \sum_{n_1+n_2=n} \frac{(i \cos \theta e^{i\frac{\pi}{2}e^{i\theta}})^{n_1} \Gamma(\sigma + n_1 + 1) 2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}}{n_1!}
$$

$$
= \sum_{n_1+n_2=n} \frac{(-e^{i\theta} \cos \theta)^{n_1} \Gamma(\sigma + n_1 + 1) 2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}}{n_1!}.
$$

(3.21)

If $n = 0$, then we have

$$
\sum_{n_1+n_2=n} \frac{(-e^{i\theta} \cos \theta)^{n_1} \Gamma(\sigma + n_1 + 1) 2^{\sigma+n_1+1}C_{\sigma+n_1+1}^{\sigma+n_1+1}}{n_1!} = 2^{\sigma+1} \Gamma(\sigma + 1)
$$

(3.22)

Let $n \geq 1$. Since

$$
\Gamma(\sigma + n_1 + 1 + (n_2 - 1) + 1) = \{\sigma + n_1 + 1 + (n_2 - 1)\} \Gamma(\sigma + n_1 + (n_2 - 1))
$$

$$
= \{\sigma + n_1 + 1 + (n_2 - 1)\} \{\sigma + n_1 + (n_2 - 2)\} \Gamma(\sigma + n_1 + (n_2 - 2))
$$

$$
= \{\sigma + n_1 + (n_2 - 1)\} \cdots \{\sigma + n_1 + 1\} \Gamma(\sigma + n_1 + 1),
$$
we have
\[
\sum_{n_1 + n_2 = n} (-e^{i \theta} \cos \theta)^{n_1} \Gamma(\sigma + n_1 + 1) \frac{2^{\sigma+n_1+1} C_{n_2}^{\sigma+n_1+1}}{n_1!} = 2^{\sigma+1} \sum_{n_1 + n_2 = n} \frac{(-e^{i \theta} \cos \theta)^{n_1} \Gamma(\sigma + n_1 + 1)}{n_1!} 2^{n_1}
\]
\[
\times \frac{(\sigma + n_1 + 1)(\sigma + n_1 + 1 + 1) \cdots \{\sigma + n_1 + 1 + (n_2 - 1)\}}{n_2!}
\]
\[
= 2^{\sigma+1} \Gamma(\sigma + n_1 + 1) \sum_{n_1 + n_2 = n} \frac{(-2e^{i \theta} \cos \theta)^{n_1}}{n_1! n_2!}
\]
\[
= 2^{\sigma+1} \Gamma(\sigma + n_1 + 1) \frac{n! \Gamma(\sigma + n_1 + 1)}{n! n_1! n_2!} (-2e^{i \theta} \cos \theta)^{n_1}
\]
\[
= 2^{\sigma+1} \Gamma(\sigma + n_1 + 1) (1 - 2e^{i \theta} \cos \theta)^n
\]
\[
= 2^{\sigma+1} \Gamma(\sigma + n_1 + 1) \left(1 - e^{i \theta} (e^{i \theta} + e^{-i \theta})\right)^n
\]
\[
= 2^{\sigma+1} \Gamma(\sigma + n_1 + 1) (-e^{2i \theta})^n.
\]

From (3.21) \sim (3.23), we obtain
\[
\sum_{n_1 + n_2 = n} K_{\sigma+n_1, n_1}(\theta) 2^{\sigma+n_1+1} C_{n_2}^{\sigma+n_1+1} k^{2n_2}
\]
\[
= ik^{2n} e^{i \theta} e^{i(\theta + \frac{\pi}{2})} \sum_{n_1 + n_2 = n} \frac{(-e^{i \theta} \cos \theta)^{n_1} \Gamma(\sigma + n_1 + 1)}{n_1!} 2^{\sigma+n_1+1} C_{n_2}^{\sigma+n_1+1}
\]
\[
= ik^{2n} e^{i \theta} e^{i(\theta + \frac{\pi}{2})} 2^{\sigma+1} \frac{(\sigma + n_1 + 1)}{n!} (e^{2i \theta})^n.
\]

Now from this and (3.20) we obtain the desired formula.
\[
\Box
\]

\textbf{Proof of Lemma 3.3.} Since
\[
\Gamma(n + 1 + l + \mu) = \Pi_{j=1}^{n} (j + l + \mu) \Gamma(1 + l + \mu),
\]
one can rewrite
\[
\sum_{n_1 + n_2 = n} \frac{(-1)^{n_2} \Gamma(n + 1 + n_2 + \mu)}{n_1! n_2! \Gamma(1 + n_2 + \mu)} = \\
\sum_{n_1 + n_2 = n} \frac{(-1)^{n_2}}{n_1! n_2!} \prod_{j=1}^{n_2} (j + n_2 + \mu)
\]
\[
= \frac{1}{n_1!} \sum_{n_1 + n_2 = n} \frac{(-1)^n}{n_1! n_2!} \left( \frac{d}{dx} \right)^n \{x^{n+n_2+\mu}\} \bigg|_{x=1}
\]
\[
= \frac{1}{n!} \left( \frac{d}{dx} \right)^n \{(1-x)^n x^{n+\mu}\} \bigg|_{x=1}
\]
\[
= \frac{1}{n!} \left( \frac{d}{dx} \right)^n \{(1-x)^n\} \cdot x^{n+\mu} \bigg|_{x=1}
\]
\[
= (-1)^n.
\]

4 The complete asymptotic expansion of the indicator function

4.1. Sound-hard polygonal obstacles. First we consider Case I. We study the asymptotic behavior of the indicator function at \( t = h_D(\omega) \) as \( \tau \to \infty \). The starting point is the expression in the following proposition.

**Proposition 4.1.** The formula
\[
I_\omega(\tau, h_D(\omega)) = e^{-\tau h_D(\omega)} \int_{\partial D} u \frac{\partial v}{\partial \nu} d\sigma,
\]
(4.1)
is valid.

**Proof.** Take \( \varphi \in C^\infty_0(B_R(0)) \) such that \( \varphi = 1 \) in a neighborhood of \( \overline{D} \). Since \( 1 - \varphi \) vanishes in a neighborhood of \( \partial D \) and \( u \) is smooth in \( B_R(0) \setminus \overline{D} \), integration by parts yields
\[
0 = \int_{B_R(0) \setminus \overline{D}} (\Delta u + k^2 u)(1 - \varphi) v dx
\]
\[
= \int_{|x|=R} \frac{\partial u}{\partial r} (1 - \varphi) v d\sigma(x) - \int_{B_R(0) \setminus \overline{D}} \{\nabla u \cdot \nabla\{(1 - \varphi) v\} - k^2 u(1 - \varphi) v\} dx.
\]
Since \( 1 - \varphi = 1 \) on \( |x| = R \), we have the first formula
\[
\int_{B_R(0) \setminus \overline{D}} \{\nabla u \cdot \nabla\{(1 - \varphi) v\} - k^2 u(1 - \varphi) v\} dx = \int_{|x|=R} \frac{\partial u}{\partial r} v d\sigma(x).
\]
Since \( \partial D \) is Lipschitz and \( v \) is smooth on \( \overline{B_R(0) \setminus D} \), we have the second formula
\[
\int_{B_R(0) \setminus \overline{D}} (\nabla u \cdot \nabla v - k^2 u v) dx = \int_{|x|=R} \frac{\partial u}{\partial r} d\sigma(x) + \int_{\partial D} u \frac{\partial v}{\partial \nu} d\sigma.
\]
Note that $\nu$ is outward to $\Omega \setminus \overline{D}$. From these two formulae we obtain

$$
\int_{x = R} \left( \frac{\partial u}{\partial r} v - u \frac{\partial v}{\partial r} \right) d\sigma(x)
= - \int_{B(R(0) \setminus \overline{D}} \{ \nabla u \cdot \nabla(\varphi v) - k^2 u \varphi v \} dx + \int_{\partial D} u \frac{\partial v}{\partial \nu} d\sigma.
$$

However, since $\varphi v = 0$ on $|x| = R$, from (2.1) one has

$$
\int_{B(R(0) \setminus \overline{D}} \{ \nabla u \cdot \nabla(\varphi v) - k^2 u \varphi v \} dx = 0.
$$

\[\Box\]

In this subsection we always assume that $\omega$ is regular with respect to $D$. Let $x_0$ denote the only one point of the set $\{x \mid x \cdot \omega = h_D(\omega)\} \cap \partial D$. $x_0$ has to be a vertex of $D$. We denote by $\Theta$ the outside angle at the vertex $x_0$. $\Theta$ satisfies $\pi < \Theta < 2\pi$ since $\omega$ is regular with respect to $D$. If one chooses a sufficiently small $\eta > 0$, then one can write

$$
B_{2\eta}(x_0) \cap (B_R(0) \setminus \overline{D}) = \{x_0 + r(\cos \theta a + \sin \theta a^\perp) \mid 0 < r < 2\eta, \ 0 < \theta < \Theta\},
$$

$$
B_{\eta}(x_0) \cap \partial D = \Gamma_p \cup \Gamma_q \cup \{x_0\}
$$

where

$$
\Gamma_p = \{x_0 + r(\cos p \omega^\perp + \sin p \omega) \mid 0 < r < \eta\},
$$

$$
\Gamma_q = \{x_0 + r(\cos q \omega^\perp + \sin q \omega) \mid 0 < r < \eta\},
$$

$$
-\pi < q < p < 0,
$$

$$
a = \cos p \omega^\perp + \sin p \omega,
$$

$$
a^\perp = -\sin p \omega^\perp + \cos p \omega.
$$

Note that $\eta$ can be arbitrary small; $\det (a a^\perp) > 0$; $p, q$ and $\Theta$ satisfy the relationship $p + \Theta = 2\pi + q$.

$p, q$ denote the rotation angles of the segment with length $\eta$ on the half line directed to $\omega^\perp$ starting from $x_0$ such that the clockwise rotation of $-p, -q$ of the segment coincides with $\Gamma_p$ and $\Gamma_q$, respectively. Note that the direction of $\omega^\perp$ is chosen in such a way that the orientation of $\omega^\perp, \omega$ coincides with that of the standard basis $e_1, e_2$ of $\mathbb{R}^2$. See also Figure 1 of [9].

We set

$$
u(r, \theta) = u(x), \ x = x_0 + r(\cos \theta a + \sin \theta a^\perp).
$$

The intersection of the exterior of open disc $B_{\eta}(x_0)$ with $\partial D$ is contained in the half-plane $x \cdot \omega \leq h_D(\omega) - \delta$ for suitable $\delta = \delta(\eta)$. From this one knows that the right-hand side of (4.1) is equal to the integral on $B_{\eta}(x_0) \cap \partial D$ modulo $O(\tau e^{-\delta \tau})$:}

$$
I_\omega(\tau, h_D(\omega)) = e^{-\tau h_D(\omega)} \int_{\partial D \cap B_{\eta}(x_0)} u \frac{\partial v}{\partial \nu} d\sigma + O(\tau e^{-\delta \tau}).
$$

(4.2)
It is easy to see that

\[
v(r, \theta) = e^{\tau (h_D(\omega) + r \sin (\theta + p))} e^{i \sqrt{\tau^2 + k^2} (x_0 \cdot \omega + r \cos (\theta + p))}
\]

\[
= e^{\tau h_D(\omega)} e^{i \sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{\tau r \sin (\theta + p)} e^{i \sqrt{\tau^2 + k^2} r \cos (\theta + p)};
\]

\[
\nabla v = (\tau \omega + i \sqrt{\tau^2 + k^2} \omega^\perp) v;
\]

\[
\nu = (\sin p) \omega^\perp - (\cos p) \omega \text{ on } \Gamma_p;
\]

\[
\nu = -(\sin q) \omega^\perp + (\cos q) \omega \text{ on } \Gamma_q.
\]

Note that \( \nu \) is the unit outward normal relative to \( \Omega \setminus \overline{D} \). From these we have

\[
e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} = -(\tau \cos p - i \sqrt{\tau^2 + k^2} \sin p) e^{i \sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{\tau r \sin p} e^{i \sqrt{\tau^2 + k^2} r \cos p} \text{ on } \Gamma_p;
\]

\[
e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} = (\tau \cos q - i \sqrt{\tau^2 + k^2} \sin q) e^{i \sqrt{\tau^2 + k^2} x_0 \cdot \omega} e^{\tau r \sin q} e^{i \sqrt{\tau^2 + k^2} r \cos q} \text{ on } \Gamma_q.
\]

Here we state a fact which can be easily seen with a minor modification of the proof of the corresponding fact in the case when \( k = 0 \) given in [7] (see also [3]).

**Proposition 4.2.** Let \( \eta \) satisfy \( \eta << 1/2k \). Then, there exist numbers \( \alpha_1, \alpha_2, \ldots, \alpha_m, \ldots \) such that:

(1) for each \( s \in ]0, 2[ \)

\[
u(r, \theta) = \sum_{m=1}^{\infty} \alpha_m J_{\mu_m}(kr) \cos \mu_m \theta \text{ in } H^1(B_{\eta}(x_0) \cap (B_R(0) \setminus \overline{D}));
\]

where

\[
\mu_m = \frac{(m - 1)\pi}{\Theta};
\]

(2) as \( m \to \infty \)

\[
|\alpha_m| = O\left(\frac{\Gamma(1 + \mu_m)}{\sqrt{\mu_m}} \left(\frac{1}{\eta k}\right)^{\mu_m}\right);
\]

(3) for each \( l = 1, \ldots \) there exists a positive number \( K_l \) such that the estimates below are valid:

\[
|u(r, 0) - \sum_{m=1}^{l} \alpha_m J_{\mu_m}(kr)| \leq K_l r^{\mu_l+1}, \ 0 < r < \eta;
\]

\[
|u(r, \theta) - \sum_{m=1}^{l} \alpha_m (-1)^{m-1} J_{\mu_m}(kr)| \leq K_l r^{\mu_l+1}, \ 0 < r < \eta.
\]
Fix \( l \geq 1 \). Recalling (3.1) for \( \mu = \mu_m \), from (4.2) \sim (4.4), (4.7) and (4.8) one obtains

\[
I_\omega(\tau, h_D(\omega))
= \int_{\Gamma_p} \sum_{m=1}^l \alpha_m J_{\mu_m}(kr)e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} dr + \int_{\Gamma_q} O(\tau^{\mu_{l+1}})e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} dr
+ \int_{\Gamma_q} \sum_{m=1}^l \alpha_m J_{\mu_m}(kr) \cos \Theta \mu_m e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} dr + \int_{\Gamma_q} O(\tau^{\mu_{l+1}})e^{-\tau h_D(\omega)} \frac{\partial v}{\partial \nu} dr
+ O(\tau e^{-\tau \delta})
\]

\[
= - (\tau \cos p - i \sqrt{\tau^2 + k^2 \sin p})e^{i \sqrt{\tau^2 + k^2} x_0 \omega} \sum_{m=1}^l \alpha_m I_{\mu_m}(\tau, p)
+ (\tau \cos q - i \sqrt{\tau^2 + k^2 \sin q})e^{i \sqrt{\tau^2 + k^2} x_0 \omega} \sum_{m=1}^l \alpha_m \cos \Theta \mu_m I_{\mu_m}(\tau, q)
+ O(\tau \int_0^\pi r^{\mu_{l+1}} e^{\tau r \sin p} dr) + O(\tau \int_0^\pi r^{\mu_{l+1}} e^{\tau r \sin q} dr) + O(\tau e^{-\tau \delta}).
\]

Note that, for \( \theta = p, q (-\pi < q < p < 0) \)

\[
0 \leq \tau \int_0^\pi r^{\mu_{l+1}} e^{\tau r \sin \theta} dr = \tau^{\mu_{l+1}} \int_0^\pi w^{\mu_{l+1}} e^w \sin \theta dw
\leq \tau^{\mu_{l+1}} \int_0^\infty w^{\mu_{l+1}} e^w \sin \theta dw = O(\tau^{-\mu_{l+1}}).
\]

Then (4.9) becomes

\[
I_\omega(\tau, h_D(\omega)) = -e^{i \sqrt{\tau^2 + k^2} x_0 \omega} \sum_{m=1}^l \alpha_m A_m(\tau) + O\left(\frac{1}{\tau^{\mu_{l+1}}}\right)
\]

where

\[
A_m(\tau) = (\tau \cos p - i \sqrt{\tau^2 + k^2 \sin p})I_{\mu_m}(\tau, p)
+ (-1)^m (\tau \cos q - i \sqrt{\tau^2 + k^2 \sin q})I_{\mu_m}(\tau, q)
\]

and \( A_m(\tau) \) is independent on \( u \). Note that we have used the relationship \( \Theta \mu_m = (m-1)\pi \). Here we introduce the new parameter given by (1.5) and make use of the crucial formula (3.2) for \( \mu = \mu_m \). Then, from (4.10) we obtain the complete asymptotic expansion of the indicator function as \( s \to \infty \).

**Theorem 4.1.** As \( s \to \infty \) the formula

\[
I_\omega(\tau, h_D(\omega))e^{-i \sqrt{\tau^2 + k^2} x_0 \omega} \sim -i \sum_{m=2}^\infty \frac{e^{i \sqrt{\tau^2 + k^2} x_0 \omega} \alpha_m e^{i \mu m}}{s^{\mu_m}}
\]

is valid.
4.2. Sound-hard piecewise linear cracks

Next we consider Case II. The starting point is the expression whose proof is essentially same as that of Proposition 4.1. We omit the description of the proof.

Proposition 4.3. The formula

\[ I_\omega(\tau, h_\Sigma(\omega)) = e^{-\tau h_\Sigma(\omega)} \int_{\Sigma} [u] \frac{\partial v}{\partial \nu} \, d\sigma, \] (4.12)

is valid where \([u] = u^+|_{\partial D} - u^-|_{\partial D} \).

The case divided into two subcases.

First consider the case when every end points of \(\Sigma_1, \Sigma_2, \ldots, \Sigma_m\) satisfies \(x \cdot \omega < h_\Sigma(\omega)\). Then \(x_0 \in \Sigma\) with \(x_0 \cdot \omega = h_\Sigma(\omega)\) should be a vertex of \(D\) and a point where two segments in some \(\Sigma_j\) meet. This means \(\Gamma_\rho\) and \(\Gamma_q\) in Subsection 4.1 should be a part of \(\Sigma_j\). Then we take the completely same polar coordinates as those of Subsection 3.1.

The same comment as that on Proposition 4.2 works for the following.

Proposition 4.4. Let \(\eta\) satisfy \(\eta << 1/2k\). Then, there exist two sequences \(\alpha^+_1, \alpha^+_2, \ldots, \alpha^+_m, \ldots\) and \(\alpha^-_1, \alpha^-_2, \ldots, \alpha^-_m, \ldots\) such that

1. For each \(s \in [0, 2]\)

\[ u^+(r, \theta) = \sum_{m=1}^{\infty} \alpha^+_m J_{\lambda^+_m}(kr) \cos \lambda^+_m \theta \text{ in } H^1(B_{s\eta}(x_0) \cap (B_R(0) \setminus \overline{D})), \]

\[ u^-(r, \theta) = \sum_{m=1}^{\infty} \alpha^-_m J_{\lambda^-_m}(kr) \cos \lambda^-_m \theta \text{ in } H^1(B_{s\eta}(x_0) \cap D) \]

where

\[ \lambda^+_m = \frac{(m-1)\pi}{\Theta}, \quad \lambda^-_m = \frac{(m-1)\pi}{2\pi - \Theta}; \]

2. As \(m \to \infty\)

\[ |\alpha^+_m| = O\left(\frac{\Gamma(1 + \lambda^+_m)}{\sqrt{\lambda^+_m}} \left(\frac{1}{\eta k}\right)^{\lambda^+_m}\right), \]

\[ |\alpha^-_m| = O\left(\frac{\Gamma(1 + \lambda^-_m)}{\sqrt{\lambda^-_m}} \left(\frac{1}{\eta k}\right)^{\lambda^-_m}\right); \]

3. For each \(l = 1, \ldots\) there exists a positive number \(K_l\) such that the estimates below are valid:

\[ |u^+(r, 0) - \sum_{m=1}^{l} \alpha^+_m J_{\lambda^+_m}(kr)| \leq K_l r^{\lambda^+_l}, \quad 0 < r < \eta; \] (4.13)

\[ |u^-(r, 2\pi) - \sum_{m=1}^{l} \alpha^-_m (-1)^{m-1} J_{\lambda^-_m}(kr)| \leq K_l r^{\lambda^-_l}, \quad 0 < r < \eta; \] (4.14)

\[ |u^+(r, \Theta) - \sum_{m=1}^{l} \alpha^+_m (-1)^{m-1} J_{\lambda^+_m}(kr)| \leq K_l r^{\lambda^+_l}, \quad 0 < r < \eta; \] (4.15)

\[ |u^-(r, \Theta) - \sum_{m=1}^{l} \alpha^-_m J_{\lambda^-_m}(kr)| \leq K_l r^{\lambda^-_l}, \quad 0 < r < \eta. \] (4.16)
A combination of (4.13) and (4.14) gives

\[
[u]_p = u^+(r, 0) - u^-(r, 2\pi)
\]

\[
= \sum_{m=1}^{l} \{ J_{\lambda_m^+}(kr)\alpha_m^+ + J_{\lambda_m^-}(kr)\alpha_m^-(-1)^m \} + O(r^{\lambda_{l+1}^+}).
\]

A combination of (4.15) and (4.16) gives

\[
[u]_q = u^+(r, \Theta) - u^-(r, \Theta)
\]

\[
= - \sum_{m=1}^{l} \{ J_{\lambda_m^+}(kr)\alpha_m^+(-1)^m + J_{\lambda_m^-}(kr)\alpha_m^- \} + O(r^{\lambda_{l+1}^+}).
\]

Note that \( \lambda_m^+ < \lambda_m^- \).

Using the same argument as that of derivation of (4.2), we have, for a positive constant \( C \) independent of \( \tau \)

\[
I_\omega(\tau, h \Sigma(\omega)) = e^{-\tau h \Sigma(\omega)} \int_{\Gamma_p} [u]_p \frac{\partial v}{\partial \nu} d\sigma + e^{-\tau h \Sigma(\omega)} \int_{\Gamma_q} [u]_q \frac{\partial v}{\partial \nu} d\sigma + O(\tau e^{-C\tau}).
\]

Then, from (4.17) and (4.18) we have, for each \( l = 1, \ldots \) as \( \tau \to \infty \)

\[
I_\omega(\tau, h \Sigma(\omega))
\]

\[
= \sum_{m=1}^{l} \alpha_m^+ \{ e^{-\tau h \Sigma(\omega)} \int_{\Gamma_p} J_{\lambda_m^+}(kr) \frac{\partial v}{\partial \nu} d\sigma - (-1)^m e^{-\tau h \Sigma(\omega)} \int_{\Gamma_q} J_{\lambda_m^+}(kr) \frac{\partial v}{\partial \nu} d\sigma \}
\]

\[
+ \sum_{m=1}^{l} \alpha_m^- \{ (-1)^m e^{-\tau h \Sigma(\omega)} \int_{\Gamma_p} J_{\lambda_m^-}(kr) \frac{\partial v}{\partial \nu} d\sigma - e^{-\tau h \Sigma(\omega)} \int_{\Gamma_q} J_{\lambda_m^-}(kr) \frac{\partial v}{\partial \nu} d\sigma \} + O \left( \frac{1}{\tau^{\lambda_{l+1}^+}} \right).
\]

Recalling (3.1) for \( \mu = \lambda_{m}^\pm \), from (4.3) and (4.4) one can write

\[
e^{-\tau h \Sigma(\omega)} \int_{\Gamma_p} J_{\lambda_m^\pm}(kr) \frac{\partial v}{\partial \nu} d\sigma = -e^{i\sqrt{\tau^2 + k^2} x_0^\pm} (\tau \cos p - i\sqrt{\tau^2 + k^2} \sin p) I_{\lambda_m^\pm}(\tau, p);
\]

\[
e^{-\tau h \Sigma(\omega)} \int_{\Gamma_q} J_{\lambda_m^\pm}(kr) \frac{\partial v}{\partial \nu} d\sigma = e^{i\sqrt{\tau^2 + k^2} x_0^\pm} (\tau \cos q - i\sqrt{\tau^2 + k^2} \sin q) I_{\lambda_m^\pm}(\tau, q).
\]

Here we introduce the new parameter \( s \) given by (1.5). Then, from (3.2) for \( \mu = \lambda_{m}^\pm \)
together with (4.19), (4.20) and (4.21) we obtain

\[ I_\omega(\tau, h\Sigma(\omega))e^{-i\sqrt{\tau^2+k^2}x_0\omega} \]

\[ = - \sum_{m=1}^{l} \alpha_m^+ \{(\tau \cos p - i\sqrt{\tau^2+k^2} \sin p)I_{\lambda_m^+}(\tau, p) + (-1)^m(\tau \cos q - i\sqrt{\tau^2+k^2} \sin q)I_{\lambda_m^-}(\tau, q)\} \]

\[ - \sum_{m=1}^{l} \alpha_m^- \{(-1)^m(\tau \cos p - i\sqrt{\tau^2+k^2} \sin p)I_{\lambda_m^-}(\tau, p) + (\tau \cos q - i\sqrt{\tau^2+k^2} \sin q)I_{\lambda_m^+}(\tau, q)\} \]

\[ + O(\frac{1}{s^{l+1}}) \]

\[ = - \sum_{m=1}^{l} \alpha_m^+ \{ie^{i(p+\frac{\pi}{2})}\lambda_m^+ k_{\lambda_m^+} + (-1)^m \frac{ie^{i(q+\frac{\pi}{2})}\lambda_m^- k_{\lambda_m^-}}{s^{l+1}}\} \]

\[ - \sum_{m=1}^{l} \alpha_m^- \{(-1)^m \frac{ie^{i(p+\frac{\pi}{2})}\lambda_m^- k_{\lambda_m^-}}{s^{l+1}} + \frac{ie^{i(q+\frac{\pi}{2})}\lambda_m^+ k_{\lambda_m^+}}{s^{l+1}}\} \]

\[ + O(\frac{1}{s^{l+1}}) \]

\[ = -i \sum_{m=1}^{l} \frac{e^{i\frac{\pi}{2}\lambda_m^+} k_{\lambda_m^+} \alpha_m^+ \{e^{ip\lambda_m^+} + (-1)^m e^{iq\lambda_m^+}\}}{s^{l+1}} \]

\[ -i \sum_{m=1}^{l} \frac{e^{i\frac{\pi}{2}\lambda_m^-} k_{\lambda_m^-} \alpha_m^- \{(-1)^m e^{ip\lambda_m^-} + e^{iq\lambda_m^-}\}}{s^{l+1}} \]

\[ + O(\frac{1}{s^{l+1}}). \]

Here from the relationship \( \Theta + p = 2\pi + q \), we see the cancellation of the coefficient of the expansion that contains \( \alpha_m^- \):

\[ (-1)^m e^{ip\lambda_m^-} + e^{iq\lambda_m^-} \]

\[ = e^{iq\lambda_m^-} \{e^{i(p-q)\lambda_m^-} + (-1)^m\}(-1)^m \]

\[ = e^{iq\lambda_m^-} \{e^{i(2\pi-\Theta)\frac{(m-1)}{2\pi-\Theta}} + (-1)^m\}(-1)^m \]

\[ = e^{iq\lambda_m^-} \{e^{i(m-1)\pi} + (-1)^m\}(-1)^m \]

\[ = e^{iq\lambda_m^-} \{-1^{m-1} + (-1)^m\}(-1)^m = 0. \]

Therefore we obtain
Theorem 4.2. As $s \to \infty$ the formula
\[ I_\omega(\tau, h_\Sigma(\omega)) e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega} \sim -i \sum_{m=2}^{\infty} e^{i \frac{\pi}{2} \lambda_m^+} k \frac{\lambda_m^+}{s^{\lambda_m^+}} \alpha_m^+ \{ e^{i p \lambda_m^+} + (-1)^m e^{i q \lambda_m^+} \}, \]
is valid.

This formula is similar to that of Theorem 4.1. The coefficients of $u^{-}$ do not appear in the expansion.

Finally we lightly comment on the case when there is an end point of some $\Sigma_j$ such that $x \cdot \omega = h \Sigma(\omega)$. Since $\omega$ is regular with respect to $\Sigma$, $x_0 \in \Sigma$ with $x_0 \cdot \omega = h \Sigma(\omega)$ should be just the point. Then, it is not difficult to prove the following proposition and theorem which correspond to the case when $p = q$ in Proposition 4.2, Theorem 4.1, respectively.

Proposition 4.5. Let $\eta$ satisfy $\eta << 1/2k$. Then, there exists a sequence $\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots$ such that:

1. for each $s \in ]0, 2[$
\[ u(r, \theta) = \sum_{m=1}^{\infty} \alpha_m J_{\frac{m-1}{2}}(kr) \cos \frac{m-1}{2} \theta \text{ in } H^1(B_{s\eta}(x_0) \setminus \Sigma); \]

2. as $m \to \infty$
\[ |\alpha_m| = O \left( \frac{\Gamma(1 + (m - 1)/2)}{\sqrt{(m - 1)/2}} \left( \frac{1}{\eta k} \right)^{(m-1)/2} \right); \]

3. for each $l = 1, \ldots$ there exists a positive number $K_l$ such that the estimates below are valid:
\[ |u(r, 0) - \sum_{m=1}^{l} \alpha_m J_{\frac{m-1}{2}}(kr)| \leq K_l r^{l/2}, \quad 0 < r < \eta; \]
\[ |u(r, 2\pi) - \sum_{m=1}^{l} \alpha_m (-1)^m J_{\frac{m-1}{2}}(kr)| \leq K_l r^{l/2}, \quad 0 < r < \eta. \]

Theorem 4.3. As $s \to \infty$ the formula
\[ I_\omega(\tau, h_\Sigma(\omega)) e^{-i\sqrt{\tau^2 + k^2} x_0 \cdot \omega} \sim -2i \sum_{n=1}^{\infty} e^{i \frac{\pi}{2} \frac{(2n-1)}{2}} k^{\frac{2n-1}{2}} \frac{\alpha_{2n} e^{i p \frac{2n-1}{2}}}{s^{\frac{2n-1}{2}}}, \]
is valid.

5 Proof of Theorem 2.1 and comment on that of Theorem 2.2

Theorem 2.1 is a consequence of Proposition 4.2, Theorem 4.1 and the following lemma.

Lemma 5.1. There exists $m \geq 2$ such that
\[ \alpha_m \{ e^{i p \mu_m} + (-1)^m e^{i q \mu_m} \} \neq 0. \]
Proof. The argument is based on that of Lemma 2 ([9]). Assume that, for all \( m \geq 2 \)
\[
\alpha_m \{ e^{ip\mu_m} + (-1)^m e^{iq\mu_m} \} = 0. \tag{5.1}
\]
If \( \Theta/\pi \) is irrational, then we know that, \( e^{ip\mu_m} + (-1)^m e^{iq\mu_m} \neq 0 \) for all \( m \neq 1 \) and thus (5.1) gives \( \alpha_m = 0 \). From Proposition 4.2 one concludes \( u = \alpha_1 J_0(k r) \) in \( 0 < r < 2\eta, 0 < \theta < \Theta \).
Then the unique continuation theorem for the Helmholtz equation says that \( u \) coincides with \( \alpha_1 J_0(k|x-x_0|) \) outside \( D \). Since \( \nabla \{ J_0(k|x-x_0|) \} \rightarrow 0 \) as \( |x| \rightarrow \infty \), from (2.2) one has \( |\nabla e^{ikx-d}| \rightarrow 0 \) as \( |x| \rightarrow \infty \). However, this is impossible. If \( \Theta/\pi \) is rational, then one can write
\[
\frac{\Theta}{\pi} = 1 + \frac{b}{a}
\]
where both \( a(\geq 2) \) and \( b(\geq 1) \) are natural numbers and satisfy \( (a,b) = 1 \). Then we find
\[
\{ m \geq 2 \mid e^{ip\mu_m} + (-1)^m e^{iq\mu_m} = 0 \} = \{ 1 + l(a+b) \mid l = 1, 2, \cdots \}.
\]
Therefore, for \( m \geq 2 \) with \( m \neq 1 + l(a+b) \), from (5.1) one concludes \( \alpha_m = 0 \). Then from (4.5) of Proposition 4.2 we have the expression
\[
|\alpha_1 + l(a+b) J_0(k r) | \cos \theta, \quad 0 < r < 2\eta, 0 < \theta < \Theta
\]
By virtue of (3.11) and (4.6) one can differentiate the right-hand side termwise. Then we immediately know that
\[
\frac{\partial u}{\partial \theta} (r, \pi) = \frac{\partial u}{\partial \theta} (r, \Theta - \pi) = 0, \quad 0 < r < 2\eta.
\]
Then, a combination of the standard reflection argument (e.g. [2]) and the uniqueness of the Cauchy problem for the Helmholtz equation yields that \( \partial u/\partial \nu_p = 0 \) on the half line starting at \( x_0 \) toward infinity opposite to another endpoint of \( \Gamma_p \); \( \partial u/\partial \nu_q = 0 \) on the half line starting at \( x_0 \) toward infinity opposite to another endpoint of \( \Gamma_q \). Here \( \nu_p \) and \( \nu_q \) denote the unit vector normal to \( \Gamma_p \) and \( \Gamma_q \), respectively. They are linearly independent.
Then (2.2) yields that \( d \cdot \nu_p = d \cdot \nu_q = 0 \). However, this is impossible.
\( \square \)

Lemma 5.1 says that one can choose
\[
m^* = \min \{ m \geq 2 \mid \alpha_m \{ e^{ip\mu_m} + (-1)^m e^{iq\mu_m} \} \neq 0 \}. \tag{5.2}
\]
Then from Theorem 4.1 one obtains
\[
\lim_{s \to -\infty} s^m |I_\omega(\tau, h_D(\omega))| = k^m |\alpha_m \{ e^{ip\mu_m^*} + (-1)^m e^{iq\mu_m^*} \}|.
\]
Since \( s \sim 2\tau \) as \( \tau \to \infty \), one has immediately
\[
\lim_{\tau \to -\infty} \tau^{m^*} |I_\omega(\tau, h_D(\omega))| = (\frac{k}{2})^{m^*} |\alpha_m \{ e^{ip\mu_m^*} + (-1)^m e^{iq\mu_m^*} \}|. \tag{5.3}
\]
From this, Theorem 4.1 and the trivial identity
\[
I_\omega(\tau, t) = e^{-\tau(t-h_D(\omega))} I_\omega(\tau, h_D(\omega)),
\]
one automatically obtains (2.4) and other conclusions. This completes the proof of Theorem 2.1.

**Remark 5.1.** We point out that, from Theorem 4.1 one has the formula

\[
\lim_{\tau \to \infty} |I_\omega(\tau, h_D(\omega))| (2\tau)^{\pi/\Theta} = |\alpha_2| k^\pi/\Theta |e^{ip\pi/\Theta} + e^{iq\pi/\Theta}|. 
\]

From the proof of Lemma 5.1 one knows that \(|e^{ip\pi/\Theta} + e^{iq\pi/\Theta}| \neq 0\). Therefore, if \(\alpha_2 \neq 0\), then (5.4) ensures the validity of (2.4). However, if \(\alpha_2 = 0\), from (5.4) one can not obtain (2.4). Here is the importance of obtaining the complete asymptotic expansion of the indicator function. The same remark works for Theorem 2.2.

**Comment on the proof of Theorem 2.2.**

The case when every end points of \(\Sigma_1, \Sigma_2, \ldots, \Sigma_m\) satisfies \(x \cdot \omega < h_{\Sigma}(\omega)\), the proof is almost parallel to that of Theorem 2.1 by virtue of Theorem 4.2. The proof in the case when there is an end point \(x_0\) of some \(\Sigma_j\) such that \(x_0 \cdot \omega = h_{\Sigma}(\omega)\) is almost similar to that of [12].

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**References**


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