Recent Development of Probe and Enclosure Methods

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7 April 2007

Abstract

The probe and enclosure methods are general ideas to extract information about unknown discontinuity (obstacles, inclusions, cracks etc.) embedded in a known background medium from the data given by the far field pattern of the scattered field, the Dirichlet-to-Neumann map or its partial knowledge at the boundary of the medium. This paper reports some recent applications of the enclosure method to several inverse problems for Laplace, heat equations and a system of equations in the linear theory of elasticity. A result related to the foundation of the probe method is also reported.

AMS: 35R30

KEY WORDS: enclosure method, probe method, crack, Mittag-Leffler function, Dirichlet-to-Neumann map, Laplace equation, exponentially growing solution, heat equation, elasticity, Carleman function, inverse scattering problem, Helmholtz equation, fixed wave number

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1 Introduction

What is the final purpose for the researchers in inverse problems? It is to extract information about unknown objects from observation data.

One says that the mathematical study of inverse problems can be divided into some steps. The first step is: the setting of the mathematical model of the observation data and unknown objects. Since the observation data are given by some physical quantities, those are described by using solutions of partial differential equations which are called the governing equations and the unknown objects are expressed in terms of the boundary conditions, coefficients of the governing equations or the values somewhere of the solutions of the equations. We call such observation data as the idealized observation data. Usually the next step is: the study of the uniqueness and stability issues of the problems. It is known that the solutions and understanding of those problems yield knowledge about the introduction of a suitable cost function and the choice of the regularization parameter (see [12]). The unknown objects are given by the solution of a minimization problem for the cost function in a class of candidates of unknown objects. This way of studying inverse problems does not depend on the individuality of each inverse problems. However, needless to say, one has to catch the character of each inverse problems for solving uniqueness and stability problems well.

Recent mathematical study of inverse problems clarified that there is another way of studying inverse problems. That brings as the second step analytical formula to extract information about unknown objects from idealized observation data. The BC method introduced by Belishev (see [5]) is a good example. This way gives a clear explanation for the mechanism of how the information about unknown objects in the formulae can be determined from the idealized observation data. In my opinion obtaining such formulae is the dream of many inverse problems researchers and usually a quite difficult task since the correspondence between the observation data and unknown objects becomes non linear and is quite involved. Moreover this is not the terminal point of the step. Since real data contain error and noise, the computation based on the formulae breaks down on the way. This is an expression of the typical character for many inverse problems and the next step is to consider when one should stop the computation. This problem is just the regularization problem of the formulae and corresponds to the regularization of the cost function in the previous second step. We believe that this way of studying inverse problems stimulates to understand the character of each inverse problems from another point of view.

The solution of the extraction problems of information about the location and shape of unknown cracks, cavities, obstacles or inclusions which we call simply discontinuity in a given medium, from observation data has a wide range of applicability. Those are typical inverse problems appearing in engineering and medicine. One can point out that previous many mathematical works for the study of the problem were devoted to the uniqueness and stability issues. For example, one can cite [2, 14, 17, 18, 73, 76] and in particular, Isakov’s contradiction argument based on the use of singular solutions and some auxiliary orthogonality relations, is quite strong for the purpose of establishing the uniqueness results for inverse boundary value problems for elliptic/parabolic/hyperbolic equations (see [67, 68, 69, 16, 52]). However, the recent situation dramatically changed. In [23] the author discovered a general idea to extract information about the location and shape of
unknown discontinuity embedded in a known background medium from the Dirichlet-to-Neumann map on the boundary of the medium. The author named the idea the probe method. This naming is coming from the use of a virtual needle, mathematically which is realized as a special sequence needle sequence (see subsection 4.1 for the definition) of the solutions of the governing equation for the background medium. The method has been applied to several inverse boundary value problems [37, 13], inverse obstacle scattering problems [24, 25] and in [10, 15] numerical testings of the probe method have been done. The study of the probe method itself still continued and, in particular, in [49] the qualitative behaviour of the needle sequence on the needle was clarified. This yielded further knowledge [50, 51] about the previous applications [60, 9] of the probe method.

By the way, when the author introduced the probe method, someone gave a negative comment on making use of the Runge approximation property of the governing equation for the construction of needle sequences. It should be pointed out that a standard proof of the Runge approximation property suggests also how to construct the needle sequences by using minimum norm solutions of infinitely many first kind linear integral equations. However, we considered how to avoid making use of the Runge approximation property and finally introduced the enclosure method.

What is the enclosure method? It is a methodology in inverse problems for partial differential equations and was introduced by the author [29]. The method gave us how to use the exponential solution for extracting a partial information about the location of unknown discontinuity which appears as discontinuity of the coefficients of a partial differential equation or a part of the boundary of the domain of definition of the solution of the equation [64, 28, 40, 42].

It should be emphasized that the motivation for the enclosure method is to make use of or find an explicit substitution of the needle sequences. It is possible to replace the exponential solutions in the original enclosure method with other solutions. This may yield a generalization of the original enclosure method as pointed out in [35]. In [34] the author brought a special fundamental solution for the Laplace operator introduced by Yarmukhamedov [85] into an inverse boundary value problem. Using the solution, we constructed an explicit substitution in an unbounded domain of the exponential solution for the Laplace equation and extended the enclosure method to an inverse boundary value problem in an unbounded domain. In [44, 66, 39] we employed the Mittag-Leffler function for the purpose and quite recently a combination of the idea of the enclosure method and the so-called complex spherical wave constructed by Isozaki has been applied in [22].

The original enclosure method makes use of infinitely many observation data. However, in [27] the author introduced a single measurement version of the enclosure method that can be divided into three parts.

(1) Find an exponential solution of the formal adjoint of the governing equation for the background medium which is parameterized by a large parameter $\tau$ and divides the whole space into two parts: in one part the absolute value of the solution decays as $\tau \to \infty$; in another part the solution grows as $\tau \to \infty$.

(2) Construct an indicator function of independent variable $\tau$ by multiplying the governing equation of the medium by the exponential solution, integrating over the domain of definition and extracting only the integral on the known boundary of the domain.

(3) Study the asymptotic behaviour of the indicator function as $\tau \to \infty$.

There are several existing applications of the single measurement version of the enclosure
method to inverse problems for elliptic equations: Inverse Source Problem [26] whose governing equation is \( \Delta u + k^2 u = \rho(x) \chi_D(x) \); Cauchy Problem [33, 65] for the stationary Schrödinger equation \(-\Delta u + V(x)u = 0\); Electrical Impedance Tomography [30, 36, 41, 61, 62] whose governing equations are \( \Delta u = 0 \) or \( \nabla \cdot (\gamma(x)\nabla u) = 0 \); Inverse Obstacle Scattering Problem at a fixed wave number [43, 41, 47, 48] whose governing equations are \( \Delta u + k^2 u = 0 \) or \( \nabla \cdot (\gamma(x)\nabla u + k^2 \gamma(x)u) = 0 \).

The aim of this expository paper is to report

- a recent application of the enclosure method to an inverse boundary value problem for a crack [63] which is a joint work with Takashi Ohe
- recent applications of a single measurement version of the enclosure method to an inverse source problem [57] and inverse boundary value problems for the heat equations [53, 54]
- an inverse problem for the crack in an elastic body [58] which is a joint work with Hiromichi Itou
- an answer [55] to the fundamental question about the probe method: how to construct a needle sequence in a closed-form.

Finally we point out that there are five survey articles [31, 32, 38, 45, 46] for the previous results about the probe and enclosure methods. This paper can be considered as the sequel to [45, 46].

2 Crack and Enclosure Method

2.1 Inverse Crack Problem and The Mittag-Leffler function

Let \( \Sigma \) be a \((n-1)\)-dimensional closed submanifold of \( \mathbb{R}^n(n = 2, 3) \) of class \( C^0 \) with boundary. \( \Sigma \) is divided into two parts: the interior and the boundary denoted by \( \text{Int} \Sigma \) and \( \partial \Sigma \), respectively.

We assume that that there exists an open subset \( D \) with Lipschitz boundary of \( \Omega \), having finitely many connected components and satisfying the following:

\[
\begin{aligned}
\mathcal{D} & \subset \Omega; \\
\Omega \setminus \mathcal{D} & \text{ is connected;} \\
\Sigma & \subset \partial \mathcal{D}.
\end{aligned}
\]

We denote by \( \nu \) the unit outward normal relative to \( D \) unless otherwise specified. Then given \( f \in H^{1/2}(\partial \Omega) \) one can find a unique solution \( u \) in a suitable function space (details are given in [51]), of an weak formulation of the elliptic problem

\[
\begin{align*}
\Delta u & = 0 \text{ in } \Omega \setminus \Sigma, \\
\frac{\partial u}{\partial \nu} & = 0 \text{ on } \Sigma, \\
u & = f \text{ on } \partial \Omega.
\end{align*}
\]

The solution does not depend on the choice of \( D \) satisfying (*)

Define

\[
\Lambda_{\Sigma} f = \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega}.
\]
We set $\Lambda_\Sigma = \Lambda_0$ in the case when $\Sigma = \emptyset$. $\Lambda_\Sigma$ is called the Dirichlet-to-Neumann map which is also invariant for the choice of $D$ satisfying ($*$). We consider the $\Sigma$ a mathematical model of an unknown perfectly insulated crack.

In this subsection we consider the problem

**Inverse Problem 2.1.** Extract information about the shape and location of the crack $\Sigma$ from $\Lambda_\Sigma$.

This is not a uniqueness question. For the study of the uniqueness issue of the inverse problems for crack see [2, 14, 18] and references therein. We also point out the paper [7] which is an application of Kirsch’s factorization method [72] (see also [74]) to the problem and gives a test whether some $\Sigma_0$ is part of $\Sigma$ or not.

The purpose of this subsection is to give an answer to Inverse Problem 2.1 in two dimensions by combining the ideas in [44, 51].

Given $\alpha \in ]0, 1[$ let $E_\alpha(z)$ denote the Mittag-Leffler function ([4])

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}.$$  

This entire function has the following remarkable asymptotic behaviour as $|z| \to \infty$:

if $|\arg z| \leq \pi \alpha / 2$, then

$$E_\alpha(z) \sim \frac{e^{z^{1/\alpha}}}{\alpha};$$

if $\pi \geq |\arg z| > \pi \alpha / 2$, then

$$E_\alpha(z) \sim -\frac{1}{\Gamma(1-\alpha)z}.$$  

(2.2)

Let $\omega = (\omega_1, \omega_2) \in S^1$ and set $\omega^\perp = (-\omega_2, \omega_1) \in S^1$. Given $y \in \mathbb{R}^2$ define the harmonic function depending on $\tau > 0$ by the equation

$$e_\alpha^\tau(x; y, \omega) = E_\alpha(\tau \{(x - y) \cdot \omega + i(x - y) \cdot \omega^\perp\}), \ x \in \mathbb{R}^2.$$  

Let $C_y(\omega, \pi \alpha / 2)$ denote the interior of the cone about $\omega$ of opening angle $\pi \alpha / 2$ with vertex at $y$:

$$C_y(\omega, \pi \alpha / 2) = \{x \in \mathbb{R}^2 \mid (x - y) \cdot \omega > |x - y| |\omega| \cos(\theta / 2)\}.$$  

Then the asymptotic behaviour of the Mittag-Leffler function described above yields that the harmonic function $e_\alpha^\tau(x; y, \omega)$ divides the whole plane into two parts as $\tau \to \infty$:

if $x$ belongs to $C_y(\omega, \pi \alpha / 2)$, then $|e_\alpha^\tau(x; y, \omega)| \to \infty$;

if $x$ does not belong to $\overline{C_y(\omega, \pi \alpha / 2)}$, then $|e_\alpha^\tau(x; y, \omega)| \to 0$.

This suggests that the harmonic function $e_\alpha^\tau(\cdot; y, \omega)$ will play a role similar to the exponential function in the enclosure method. Let $\Sigma \subset \mathbb{R}^2$.

**Definition 2.1.** Define the indicator function in this section by the formula

$$I_{(y, \omega)}^\alpha(\tau) = \langle (\Lambda_\Sigma - \Lambda_1)(e_\alpha^\tau(\cdot; y, \omega)|_{\partial \Omega}), e_\alpha^\tau(\cdot; y, \omega)|_{\partial \Omega} \rangle.$$  

The next theorem gives us an answer to the asymptotic behaviour of the indicator function as $\tau \to \infty$ for each fixed $(y, \omega)$.

**Theorem 2.1([63]).** Given $(y, \omega) \in \Omega \times S^1$ we have:
(1) if \( \{ C_y(\omega, \pi \alpha/2) \} \cap \Sigma = \emptyset \), then

\[
\lim_{\tau \to \infty} |I^\alpha_{(y, \omega)}(\tau)| = 0;
\]

(2) if \( \{ C_y(\omega, \pi \alpha/2) \} \cap \Sigma \neq \emptyset, \{ C_y(\omega, \pi \alpha/2) \} \subset \text{Int} \Sigma \) and \( C_y(\omega, \pi \alpha/2) \cap \Sigma = \emptyset \), then

\[
\lim \inf_{\tau \to \infty} |I^\alpha_{(y, \omega)}(\tau)| > 0;
\]

(3) if \( \{ C_y(\omega, \pi \alpha/2) \} \cap \Sigma \neq \emptyset, \{ C_y(\omega, \pi \alpha/2) \} \subset \text{Int} \Sigma \) and \( C_y(\omega, \pi \alpha/2) \cap \Sigma \neq \emptyset \), then

\[
\lim_{\tau \to \infty} |I^\alpha_{(y, \omega)}(\tau)| = \infty.
\]

The validity of (1) can be easily understood, however, the proof of (2) and (3) are not trivial. It is based on the idea introduced in [51].

Theorem 2.1 yields a new uniqueness theorem for some partial information about unknown cracks with infinitely many boundary data.

For the description we introduce a concept of visibility (see also [66]).

**Definition 2.2.** We say that a point in \( \Omega \) is visible if it can be connected with infinity by a straight line without intersecting \( \Sigma \). We denote by \( V(\Sigma) \) the set of all points in \( \Omega \) that are visible. In this paper we call the set \( \Omega \setminus V(\Sigma) \) the enclosure of \( \Sigma \).

It is easy to see that the point \( y \) in \( \Omega \) is visible if and only if there exist \( \alpha \in ]0, 1[ \) and \( \omega \in S^1 \) such that \( \{ C_y(\omega, \pi \alpha/2) \} \cap \Sigma = \emptyset \). Thus we have

\[
V(\Sigma) = \bigcup_{0<\alpha<1} \bigcup_{\omega \in S^1} \{ y \in \Omega \mid \{ C_y(\omega, \pi \alpha/2) \} \cap \Sigma = \emptyset \}.
\]

It is trivial to see that the set \( V(\Sigma) \) is a non empty open subset of \( \Omega \) and satisfies

\[
\Sigma \subset \Omega \setminus V(\Sigma).
\]

Thus knowing \( V(\Sigma) \) yields the estimation of \( \Sigma \) from above. In some cases, for example, if \( \Sigma \) is given by a single segment, then we have \( \Sigma = \Omega \setminus V(\Sigma) \). It is possible to give more complicated examples that satisfies \( \Sigma = \Omega \setminus V(\Sigma) \), however, in general, \( \Sigma \neq \Omega \setminus V(\Sigma) \).

The next theorem tells us that the asymptotic behaviour of the indicator functions \( I^\alpha_{(y, \omega)} \) for all \( \alpha \) and \( (y, \omega) \in \Omega \times S^1 \) uniquely determine set \( V(\Sigma) \) except for a finite set.

**Corollary 2.2([63]).** Assume that, for each fixed \( \alpha \in ]0, 1[ \) and \( (y, \omega) \in \Omega \times S^1 \) we have

\[
\lim_{\tau \to \infty} < (\Lambda_{\Sigma_1} - \Lambda_{\Sigma_2}) e^\alpha(\cdot; y, \omega)|_{\partial \Omega}, e^\alpha(\cdot; y, \omega)|_{\partial \Omega} >= 0.
\]

Then \( V(\Sigma_1) \setminus \partial \Sigma_2 = V(\Sigma_2) \setminus \partial \Sigma_1 \).

Note that we do not assume that \( \Lambda_{\Sigma_1} = \Lambda_{\Sigma_2} \). This is different from the uniqueness theorem for the full information about the cracks given by Eller [14].

In [63] an algorithm based on these theoretical investigation has been introduced and tested.
2.2 Reconstruction of A Linear Crack in An Elastic Body

In this subsection we present an application of the single measurement version of the enclosure method to an inverse crack problem for an elastic body. By $u = (u_i)_{i=1,2,3}$, $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ we denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for a *homogeneous isotropic* material consist of the constitutive law (Hooke’s law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \quad i, j = 1, 2, 3 \tag{2.3}$$

and the equilibrium conditions without any body forces

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad i, j = 1, 2, 3. \tag{2.4}$$

Here and in what follows we use the summation convention. The $\lambda$ and $\mu$ are Lamé constants, $\delta_{ij}$ is the Kronecker’s delta and the strain-displacement relation is given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \quad u_{ij} = \partial_j u_i, \quad i, j = 1, 2, 3. \tag{2.5}$$

In the state of a *plane strain* (see [21]), the 3rd component $u_3$ of the displacement $u$ is zero, while the components $u_1$ and $u_2$ are functions of $x_1$ and $x_2$ only, hence $\varepsilon_{i3} = 0$, $\sigma_{13} = \sigma_{23} = 0$.

Let $\Omega$ be a bounded convex domain in $\mathbb{R}^2$, representing a homogeneous elastic plate. Then (2.3), (2.4) and (2.5) give the system of equations

$$A(\partial_x) u = 0$$

for $u = (u_1, u_2)^T$, where the superscript T denote matrix transposition and $A(\partial_x) = A\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$,

$$A(\xi_1, \xi_2) = \begin{pmatrix} \mu \xi_1^2 + (\lambda + \mu) \xi_1 \xi_2 & (\lambda + \mu) \xi_1 \xi_2 \\ (\lambda + \mu) \xi_1 \xi_2 & \mu \xi_2^2 + (\lambda + \mu) \xi_2^2 \end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2. \tag{2.6}$$

We assume that shearing strain $\mu > 0$, modulus of compression $\lambda + \mu > 0$, in which case it is easy to see that the operator $A$ is elliptic. Moreover we introduce the boundary stress operator $T(\partial_x) = T\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ defined by

$$T(\xi_1, \xi_2) = \begin{pmatrix} (\lambda + 2\mu) \nu_1 \xi_1 + \mu \nu_2 \xi_2 & \mu \nu_2 \xi_1 + \lambda \nu_1 \xi_2 \\ \lambda \nu_2 \xi_1 + \mu \nu_1 \xi_2 & \mu \nu_1 \xi_1 + (\lambda + 2\mu) \nu_2 \xi_2 \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^T$ is the unit outward normal to $\partial \Omega$.

We define the *internal energy density* by

$$E(u, v) = \frac{1}{2} \left\{ (\lambda + 2\mu)(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) + \lambda(u_{1,1}v_{2,2} + u_{2,2}v_{1,1}) + \mu(u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1}) \right\}. \tag{2.7}$$
Then it is easy to see that $E(u, u)$ is a non-negative quadratic form and that $E(u, u) = 0$ if and only if $u$ is a rigid displacement

$$u = (k_1 + k_0x_2, k_2 - k_0x_1)^T$$

with arbitrary constants $k_0$, $k_1$ and $k_2$. It is easily seen that

$$F_1 = (1, 0)^T, \quad F_2 = (0, 1)^T, \quad F_3 = (x_2, -x_1)^T = (1, 1)^T \wedge x$$

consist of a basis of the space $\mathcal{F}$ of such rigid displacements.

We denote by $\gamma$, which is the straight line segment $PQ$ for any $P \in \partial \Omega$ and $Q \in \Omega$ the crack in $\overline{\Omega}$. Let $Q'$ be a point of intersection of an extension of the crack $PQ$ and $\partial \Omega$, $\gamma'$ denotes $PQ'$. Then, $\Omega$ is divided into two parts $\Omega_+$ and $\Omega_-$ by $\gamma'$.

Let $H^1(\Omega_{\pm})$ be the space of the restrictions to $\Omega_{\pm}$ of all $u \in H^1(\mathbb{R}^2)$, respectively. We denote by $\pi^\pm$ the operators of restriction from $\mathbb{R}^2$ or $\Omega \setminus \gamma$ to $\Omega_{\pm}$, respectively. It is well known that any bounded convex set has Lipschitz boundary (e.g., see [20]). Therefore both $\Omega_+$ and $\Omega_-$ are Lipschitz domains. Let $\hat{\pi}^\pm$ be the (continuous) trace operators from $H^1(\Omega_{\pm})$ to $H^{\frac{1}{2}}(\partial \Omega_{\pm})$, respectively. For any $u$ defined in $\Omega \setminus \gamma$ we write $u_{\pm} = \pi^\pm u$.

Let $H^1(\Omega \setminus \gamma)$ be the space of all $u \in L^2(\Omega \setminus \gamma)$ such that $u_+ \in H^1(\Omega_+)$, $u_- \in H^1(\Omega_-)$ and $\hat{\pi}^+ u_+ |_{\gamma \setminus \gamma} = \hat{\pi}^- u_- |_{\gamma \setminus \gamma}$. The norm in $H^1(\Omega \setminus \gamma)$ is defined by

$$\|u\|_{H^1(\Omega \setminus \gamma)}^2 = \|u_+\|_{H^1(\Omega_+)}^2 + \|u_-\|_{H^1(\Omega_-)}^2.$$

We emphasize that the traces of $u \in H^1(\Omega \setminus \gamma)$ on opposite sides of $\gamma$ may be distinct (i.e. $\hat{\pi}^+ u_+ |_{\gamma} \neq \hat{\pi}^- u_- |_{\gamma}$).

**Definition 2.3.** Let $\Gamma_D$ be an arbitrary nonempty open subset of $\partial \Omega \setminus \gamma'$ with $P \notin \Gamma_D$. We say that for given $g \in L^2(\partial \Omega \setminus \Gamma_D)$ a $u \in H^1(\Omega \setminus \gamma)$ is a weak solution of the problem

$$Au(x) = 0 \text{ in } \Omega \setminus \gamma,$$

$$Tu(x) = 0 \text{ on } \gamma^\pm,$$

$$u(x) = 0 \text{ on } \Gamma_D,$$

$$Tu(x) = g \text{ on } \partial \Omega \setminus \Gamma_D$$

if the trace of $u$ onto $\Gamma_D$ vanishes and

$$2 \int_{\Omega \setminus \gamma} E(u, v) \, dx = \int_{\partial \Omega \setminus \Gamma_D} g^T v \, ds$$

for arbitrary $v \in H^1(\Omega \setminus \gamma)$ with $v = 0$ on $\Gamma_D$.

Using a similar argument to [8], one can prove the problem (2.6) has a unique weak solution.

However, note that in the case that $\Gamma_D = \phi$ the weak solution of (2.6) does not always exist. For existence of $u$ the necessary and sufficient compatibility condition on the data is the orthogonality of the right hand side of (2.7) to the rigid displacements meaning

$$\int_{\partial \Omega} g^T Fk \, ds = 0.$$
When this orthogonality condition (2.8) is fulfilled, one can prove that there exists weak solution of (2.6) \( u \in H^1(\Omega \setminus \gamma) \), using a similar argument to [8]. In the case that \( \Gamma_D = \phi \), any two weak solutions differ by an arbitrary rigid displacement \( Fk \).

Let \( u \) be a weak solution of (2.6). We define the bounded linear functional on the Banach space

\[
H^\frac{1}{2}(\Gamma_D) = \{ f |_{\Gamma_D} \mid f \in H^\frac{1}{2}(\partial\Omega) \}
\]

with norm \( \|g\| = \inf \{ \|f\|_{H^\frac{1}{2}(\partial\Omega)} \mid f = g \text{ on } \Gamma_D \} \) for each \( g \in H^\frac{1}{2}(\Gamma_D) \) as follows. Given \( h \in H^\frac{1}{2}(\Gamma_D) \) choose \( f \in H^\frac{1}{2}(\partial\Omega) \) in such a way that \( h = f \) on \( \Gamma_D \). Next choose \( v \in H^1(\Omega) \) in such a way that \( f = v|_{\partial\Omega} \).

Define

\[
\langle Tu, h \rangle \equiv 2 \int_{\Omega \setminus \gamma} E(u, v) \, dx - \int_{\partial\Omega \setminus \Gamma_D} g^T v \, ds.
\]

From the definition of the weak solution it follows that the value \( \langle Tu, h \rangle \) does not depend on the choices \( f \) and \( v \). Then a trace theorem yields that \( \langle Tu, \cdot \rangle \) is bounded linear functional on \( H^\frac{1}{2}(\Gamma_D) \).

**Inverse Problem 2.2.** Find \( \gamma \) from the single Cauchy data on \( \partial\Omega \), which means \( u|_{\partial\Omega \setminus \Gamma_D} \), \( g \) and \( \langle Tu, \cdot \rangle \) of a weak solution \( u \). This problem can be applied to a nondestructive testing.

Here we report that one can apply the enclosure method to this problem.

Let \( S^1 \) denote the unit circle. We now define the support function \( h_\gamma \) of \( \gamma \) by

\[
h_\gamma(\omega) = \sup_{x \in \gamma} x \cdot \omega, \quad \omega \in S^1.
\]

Note that the line \( x \cdot \omega = h_\gamma(\omega) \) is independent of the origin of coordinates chosen, although the value of \( h_\gamma(\omega) \) depends on it.

It is well known that there exists a direction \( \vartheta \in S^1 \) such that, for all \( x \in \Omega \) \( (x - P) \cdot \vartheta > 0 \). Let \( \omega \in S^1 \) satisfy \( \omega \cdot \vartheta > 0 \). We say that \( \omega \) is regular with respect to \( \gamma \) if the intersection of the line \( x \cdot \omega = h_\gamma(\omega) \) with \( \gamma \) is just given by the tip \( Q \).

It is clear that given \( \omega \) with \( \omega \cdot \vartheta > 0 \), is regular with respect to \( \gamma \) if and only if \( \omega \) is not parallel to the unit normal of \( \gamma \).

Let \( \omega \in S^1 \) and take \( \omega ^\perp \subset S^1 \) perpendicular to \( \omega \) satisfying \( \det (\omega ^\perp, \omega ) > 0 \). Following [27], for \( \tau > 0 \) we define a function \( v \) satisfying \( Av = 0 \) in \( \mathbb{R}^2 \) by

\[
v = (\omega + i \omega ^\perp)e^{\tau x(\omega + i \omega ^\perp)}.
\]

Using this function, we define a mathematical indicator.

**Definition 2.4.** Let \( u \) be a weak solution of (2.6). Define

\[
I_\omega(\tau, t) = e^{-\tau t} \left\{ \int_{\partial\Omega \setminus \Gamma_D} g^T v \, ds + \langle Tu, v|_{\Gamma_D} \rangle - \int_{\partial\Omega \setminus \Gamma_D} u^T T v \, ds \right\}
\]

for \( \tau > 0 \) and \( t \in \mathbb{R} \).

For the description of the condition on the surface traction applied to the elastic body we prepare the concept of the well-controlled surface traction.
Definition 2.5. We say that the surface traction \( g \) is well controlled if there exists \( k \neq 0 \) satisfying
\[
\int_{\partial \Omega^n \setminus \gamma} g^T F k \, ds \neq 0.
\]
One may think that the concept of well-controlled traction depends on the unknown crack. However, we can give an example of well-controlled traction which is independent of the unknown crack. See [58] for the detail of the example. Besides if the direction of the unknown crack is a priori known, then the concept of well-controlled traction is independent of the crack.

The following result is the extraction formula of the value of the support function of \( \gamma \).

Theorem 2.4 ([58]). Assume that \( \partial \Omega \setminus (\{P\} \cup \Gamma_D) \) is \( C^2 \). Let \( g \in C^1(\partial \Omega \setminus (\{P\} \cup \Gamma_D)) \) and be well controlled. Let \( \omega \) be regular with respect to \( \gamma \). The formula
\[
h_{\gamma}(\omega) = \lim_{\tau \to \infty} \frac{\log |I_\omega(\tau, 0)|}{\tau},
\]
is valid. Moreover we have
- if \( t \geq h_{\gamma}(\omega) \), then \( \lim_{\tau \to \infty} |I_\omega(\tau, t)| = 0 \);
- if \( t < h_{\gamma}(\omega) \), then \( \lim_{\tau \to \infty} |I_\omega(\tau, t)| = \infty \).

Two remarks are in order.
1. In the case that \( v = ((\omega + i\omega^\perp) \cdot x) \exp^{\tau x(\omega + i\omega^\perp)}(\omega + i\omega^\perp) \), Theorem 2.4 also holds.
2. In the case that \( \Gamma_D = \phi \), assuming that \( g \) is well controlled and \( u \notin \mathcal{F} \), Theorem 2.4 is also valid for both of \( v \).

For both of \( v \) the formula is direct corollary of the trivial identity
\[
I_\omega(\tau, t) = e^{\tau(h_{\gamma}(\omega) - t)} I_\omega(\tau, h_{\gamma}(\omega))
\]
and the following Lemma.

Lemma 2.5 ([58]). If \( \omega \) is regular with respect to \( \gamma \), then there exist positive constants \( \lambda = \lambda(\omega, \gamma) \) and \( M = M(\omega, \gamma) \) such that
\[
\lim_{\tau \to \infty} \tau^\lambda |I_\omega(\tau, h_{\gamma}(\omega))| = M.
\]

The proof of this lemma is divided into three parts.
First in a suitable coordinate we write the solution \( u \) down at the tip of crack in a convergent series. For the convergence of the series we employ an analytic continuation argument of a stress function due to Muskhelishvili [77] and show that the tip is a removable singularity of the continuation of the stress function in a transformed domain.
Second using the expansion in the first step, we compute the complete asymptotic expansion of the indicator function.
Third we show that if all the coefficients of the asymptotic expansion vanish, then \( g \) should no be well controlled.
Remark 2.1. Define another indicator function $\tilde{I}_\omega(\tau, t)$ by

$$\tilde{I}_\omega(\tau, t) = e^{-\tau t} \left\{ \int_{\partial\Omega\setminus\Gamma_D} g^T v \, ds - \int_{\partial\Omega\setminus\Gamma_D} u^T T v \, ds \right\}.$$ 

This indicator function can be calculated from the surface displacement and traction fields outside $\Gamma_D$. Assume that $x \cdot \omega < h_\gamma(\omega)$ for all $x \in \overline{\Gamma}_D$. Then $e^{-\tau h_\gamma(\omega)}(Tu, v|_{\Gamma_D})$ is exponentially decaying as $\tau \to \infty$. This gives $\tilde{I}_\omega(\tau, t) = I_\omega(\tau, t)$ at $t = h_\gamma(\omega)$ modulo exponentially decaying as $\tau \to \infty$. Therefore from Lemma 2.5 one concludes that Theorem 2.4 is valid for $\tilde{I}_\omega(\tau, t)$ under the assumption on $\omega$, $\gamma$ and $\Gamma_D$ mentioned above.

We have already known that our method works also for a linear crack in a homogeneous anisotropic elastic body. This will be reported in a forthcoming paper.

The next very interesting and challenging problem is to apply the enclosure method to the cavity/inclusion extraction problem in an elastic body in two dimensions from a single set of measure data. This will be our next project. Any application of the single measurement version of the enclosure method to corresponding inverse problems in three dimensions remains open.

3 Enclosure Method for The Heat Equation

This section gives some applications of the single measurement version of the enclosure method to three typical inverse problems for the heat equation. Exponential solutions for the backward heat equation play the central role.

3.1 Inverse Source Problem and Complex Exponential Solution

Let $\Omega$ be a bounded domain of $\mathbb{R}^n (n = 2, 3)$ with smooth boundary. Let $T$ be an arbitrary positive number. Let $u = u(x, t)$ satisfy

$$u_t = \Delta u + f(x, t) \text{ in } \Omega \times ]0, T[,$$

$$u(x, 0) = 0 \text{ in } \Omega.$$ 

In this subsection we consider an inverse source problem for the heat equation. The problem is

**Inverse Problem 3.1.** Assume that there exist a positive number $T_0$ less than $T$ and point $x_0 \in \Omega$ such that $f(x_0, T_0) \neq 0$ and $f(x, t) = 0$ for all $0 < t < T_0$ and all $x \in \Omega$. Extract $T_0$ and information about the set $\{x \in \Omega \mid f(x, T_0) \neq 0\}$ from the data $u|_{\partial\Omega \times ]0, T[}$, $\partial u/\partial n|_{\partial\Omega \times ]0, T[}$. The number $T_0$ and the set $\{x \in \Omega \mid f(x, T_0) \neq 0\}$ are the time and position when and where the heat source $f(x, t)$ firstly appeared.

Needless to say, one can not uniquely determine general $f(x, t)$ from a single set of temperature $u(x, t)$ and flux distributions $\partial u/\partial n(x, t)$ for $(x, t) \in \partial\Omega \times ]0, T[$. However, under some a priori assumption on the form of the unknown source one can extract the full or partial information about the source. Here we give a list of the form of unknown sources in existing related results.
1. Yamatani-Ohnaka [84] assumed that the source takes the form
\[ f(x, t) = \sum_{j=1}^{N} p_j \delta(x - x_j, t - t_j) \]
and that \( 0 < \max_j t_j < T \) and \( p_j < 0 \).

2. Yamamoto [83] assumed that the source takes the form
\[ f(x, t) = \sigma(t) g(x), \quad \sigma(0) \neq 0 \]
and that \( \sigma(t) \) is known.

3. EL Badia-Ha Duong [3] assumed that the source takes the form
\[ f(x, t) = \sum_{j=1}^{N} c_j(t) \delta(x - x_j) \]
and that there exists a known \( T < T \) such that, for all \( t \geq T \), \( c_j(t) = 0 \) and for all \( t < T \), \( c_j(t) < 0 \).

In 2 and 3 the occurring time of the source is known; in 1 and 3 it is assumed that the vanishing time occurs after a known time; in 2 the time varying part \( \sigma(t) \) is assumed to be known.

Now we state one of results in [57] in the case when \( n = 2 \). We assume that the unknown source takes the form
\[ f(x, t) = \sum_{j=1}^{N} \chi_{P_j \times [T_j, T]}(x, t) \rho_j(x, t) \]
where
- Each \( P_j \subset \Omega \) is given by the interior of a polygon
- If \( j \neq j' \), then \( P_j \cap P_{j'} = \emptyset \)
- \( T_j \) satisfies \( 0 \leq T_j < T \)
- \( \rho_j \) is Hölder continuous
- \( \rho_j(p, T_j) \neq 0 \) at all vertices of the convex hull of \( P_j \)

The exponential solution for this problem is given by the function
\[ v(x, t) = e^{-\langle z \cdot z \rangle t} e^{z \cdot z} \]
where
\[ z = c\tau \left( \omega + i \sqrt{1 - \frac{1}{c^2 \tau^2}} \omega^\perp \right), \quad \tau > e^{-2}. \]

The function \( v \) satisfies the backward heat equation \( \partial_t v + \Delta v = 0 \) and the complex vector \( z \) satisfies the equation \( z \cdot z = \tau \).

Since for each fixed \( s \) we have
\[ |e^{\tau s} v(x, t)| = e^{\tau(s - t + c x \cdot \omega)}, \]
we know that the asymptotic behaviour of \( e^{\tau s} v(x, t) \) as \( \tau \to \infty \) divides the space time into two parts \( s - t + c x \cdot \omega > 0 \) and \( s - t + c x \cdot \omega < 0 \): if \( s - t + c x \cdot \omega > 0 \), then \( |e^{\tau s} v(x, t)| \) is growing; if \( s - t + c x \cdot \omega < 0 \), then \( |e^{\tau s} v(x, t)| \) is decaying.
Using function $v$, we define the indicator function of independent variable $\tau(>c^{-2})$ by the formula

$$I_{\omega,c}(\tau; s) = e^{\tau s} \int_{0}^{T} \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) dS dt, \tau > c^{-2}.$$  

The indicator function can be calculated from a single set of temperature $u(x, t)$ and flux distributions $\partial u/\partial \nu(x, t)$ for $(x, t) \in \partial \Omega \times ]0, T[.$

For the description of the asymptotic behaviour of the indicator function we introduce some notation.

We denote by $\omega(c)$ the unit vector in three dimensions

$$\omega(c) = \frac{1}{\sqrt{c^2 + 1}} (c \omega, -1)^T.$$  

Set

$$D = \cup_{j=1}^{N} (P_j \times ]T_j, T[).$$  

and introduce the support function for $D$:

$$h_D(\vartheta) = \sup_{(x,t) \in D} (x, t)^T \cdot \vartheta, \vartheta \in S^2.$$  

**Theorem 3.1** ([57]). Assume that

- $\omega(c)$ is regular with respect to $D$
- Observation time $T$ and $c$ satisfy

$$\sup_{x \in \Omega} (x, T)^T \cdot \omega(c) < h_D(\omega(c)).$$  

Then, the formula

$$\lim_{\tau \to \infty} \frac{\log |I_{\omega,c}(\tau; 0)|}{\tau} = \sqrt{c^2 + 1} h_D(\omega(c)),$$  

is valid. Moreover we have

- If $s \leq -\sqrt{c^2 + 1} h_D(\omega(c))$, then $\lim_{\tau \to \infty} |I_{\omega,c}(\tau; s)| = 0$
- If $s > -\sqrt{c^2 + 1} h_D(\omega(c))$, then $\lim_{\tau \to \infty} |I_{\omega,c}(\tau; s)| = \infty$  

The main feature of our approach is

- it is based on a simple one line formula (3.2)
- we do not make use of the exact controllability of the heat equation nor the completeness of the eigenfunctions of the Dirichlet Laplacian in $\Omega$
- the assumptions on the unknown source is quite general
- the method provides us a brief information about the time and the location when and where the unknown source firstly appeared instead of the detailed information of the source
- it is possible to give a regularization of the formula (3.2) by employing the argument done in [36]

Here we give an interpretation of the result. The point is the interpretation of the condition (3.1). Assume that a virtual signal started at the point $x_0$ with $x_0 \cdot \omega(c) = h_D(\omega(c))$ with propagation speed $1/c$ and propagated on the plane spanned by two vectors $(c \omega, -1)^T$ and $(0, -1)^T$ to the exterior of $D$. Then condition (3.1) means that the arrival time of
the signal at the boundary of $\Omega$ is less than $T$. From this point of view condition (3.1) is quite natural and understandable. However, it should be noted that we do not know what is the signal. However, one may say that the indicator function picks up a *virtual* signal with an arbitrary fixed propagation speed caused by the unknown source. One can apply the enclosure method to the corresponding inverse source problem for the wave equation. Then changing $c$ in the inverse source problem for the heat equation

$$
\partial_t u = \Delta u + f(x, t)
$$

$$
v = e^{-\tau t} e^{x \cdot z}
$$

$$
z = c\tau (\omega + i \sqrt{1 - \frac{1}{c^2} \omega^2})
$$

$$
\tau > c^{-2}, c > 0
$$

has the same effect as changing *propagation speed* of a *real* signal in the inverse source problem for the wave equation

$$
\partial_t^2 u = \frac{1}{c^2} \Delta u + f(x, t)
$$

$$
v = e^{-\tau t} e^{x \cdot z'}
$$

$$
z' = c\tau \omega
$$

$$
c > 0.
$$

### 3.2 Inverse Source Problem and Real Exponential Solution

It should be noted that one can also extract the occurring time $\min T_j$ directly by using another exponential solution. Here we present a simple implication of the idea to Inverse Problem 3.1 which is not described in [57].

Given $\omega \in S^{n-1}$, $s \in \mathbb{R}$ define the *indicator function* $I_{\omega}(\tau; s)$ by the formula

$$
I_{\omega}(\tau; s) = e^{\tau s} \int_0^T \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) dS dt, \tau > 0
$$

where

$$
v = v(x, t) = e^{\sqrt{\tau} x \cdot \omega - \tau t}.
$$

The function $v$ takes positive values and satisfies the backward heat equation $v_t + \Delta v = 0$ in the whole space. Moreover $e^{\tau s} v$ has the special character:

- if $t > s$, then $\lim_{\tau \to -\infty} e^{\tau s} v(x, t) = 0$;
- if $t < s$, then $\lim_{\tau \to -\infty} e^{\tau s} v(x, t) = \infty$.

Note that the function $w(x) = e^{\sqrt{\tau} x \cdot \omega}$ satisfies the equation

$$
\Delta w - \tau w = 0
$$
in the whole space. In the following we make use of the asymptotic behaviour of \( w \) as \( \tau \to \infty \).

The list of the assumptions on the source \( f(x,t) \) is the following.

- \( f(x,t) \) takes the form
  \[
  f(x,t) = \chi_D(x,t)\rho(x,t)
  \]
  where \( D \subset \overline{\Omega} \times [T_0, T] \) is a Lebesgue measurable set with \( D \cap \{(x,t) | t = T_0\} = \{x_1, \ldots, x_N\} \subset \Omega \); \( \rho \) is essentially bounded on \( D \) and coincides with a uniformly H"older continuous function with exponent \( \theta_j \in ]0, 1[ \) on the intersection of \( D \) with an open neighbourhood of each point \( x_j \) of \( x_1, \ldots, x_N \).

Thus the source may appear at the time \( T_0 \) firstly and points \( x_1, \ldots, x_N \).

The last condition is ensured if \( \rho(x_i, T_0) \) for all \( i \) with \( x_i \cdot \omega = \max_{j=1, \ldots, N} x_j \cdot \omega \) have the same sign. In this case the set of all \( x \in \Omega \) with \( f(x, T_0) \neq 0 \) is contained in \( \{x_1, \ldots, x_N\} \).

Under these assumptions we obtain

**Theorem 3.2.** As \( \tau \to \infty \) we have

\[
\frac{1}{\tau} \log |I_\omega(\tau; 0)| = -T_0 + \frac{\max_{j=1, \ldots, N} x_j \cdot \omega}{\sqrt{\tau}} - \frac{(n+1)\log \tau}{\tau} + O\left(\frac{1}{\tau^{1+\theta_0}}\right)
\]

(3.3)

\[
\log |\sum_{x_i \omega = \max_{j=1, \ldots, N} x_j \omega} \rho(x_i, T_0) |W_i| n!|\]

\[
+ \frac{\rho(x_i, T_0) |W_i| n!}{\delta^n} \frac{1}{\tau} + O\left(\frac{1}{\tau^{1+\theta_0}}\right)
\]

where

\[
\theta_0 = \min \{ \theta_i | x_i \cdot \omega = \max_{j=1, \ldots, N} x_j \cdot \omega \}.
\]

Moreover, we have:

- if \( s < T_0 \), then \( \lim_{\tau \to -\infty} |I_\omega(\tau; s)| = 0 \);
- if \( s > T_0 \), then \( \lim_{\tau \to -\infty} |I_\omega(\tau; s)| = \infty \).

The formula (3.3) says that one can extract the time \( T_0 \) and \( \max_{j=1, \ldots, N} x_j \cdot \omega \) for given \( \omega \) from the data \( u|_{\partial \Omega \times [0, T]} \), \( \partial u/\partial \nu|_{\partial \Omega \times [0, T]} \). Since \( \max_{j=1, \ldots, N} x_j \cdot \omega \) is nothing but the value of the support function of the set \( \{x_1, \ldots, x_N\} \), as a corollary, we obtain the extraction formula of the convex hull of the set. Note that we do not assume that \( N \) is known.

However, it should be pointed out that the space information of unknown source in the formula (3.3) is behind the time information. In (3.2) we could obtain the space and time
information at the same time. This is an advantage of the use of complex exponential solutions.

Proof of Theorem 3.2. Since we have the trivial identity
\[ I_\omega(\tau; s) = e^{\tau(s - T_0)} I_\omega(\tau; T_0) \quad \forall s, \]
it suffices to study the asymptotic behaviour of the indicator function at \( s = T_0 \). Integration by parts gives
\[ \int_0^T \int_\Omega f(x, t)v(x, t)dxdt = \int_\Omega u(x, T)v(x, T)dx + \int_0^T \int_{\partial\Omega} \left( \frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) dSdt. \]
Thus we have the representation of the indicator function at \( s = T_0 \):
\[ I_\omega(\tau; T_0) = e^{\tau T_0} \int_0^T \int_\Omega f(x, t)v(x, t)dxdt - e^{\tau T_0} \int_\Omega u(x, T)v(x, T)dx. \quad (3.5) \]
Since \( T > T_0 \), it is easy to see that the absolute value of the second term of the right hand side is dominated by
\[ \int_\Omega |u(x, T)|dx e^{\tau sup_{x \in \Omega} x \omega} e^{-\tau(T - T_0)} = O(e^{-\tau(T - T_0)/2}) \quad (3.6) \]
as \( \tau \to \infty \).

On the other hand, from the first two assumptions on \( f \) one knows that the first term of the right hand side of (3.5) takes the form
\[ e^{\tau T_0} \int_D \rho(x, t)v(x, t)dxdt \]
\[ = e^{\tau T_0} \sum_{j=1}^N \int_{V_j} \rho(x, t)e^{\sqrt{\tau}x \omega - \tau t}dxdt + e^{\tau T_0} \int_{D \cup \cup_{j=1}^N V_j} \rho(x, t)e^{\sqrt{\tau}x \omega - \tau t}dxdt. \quad (3.7) \]
Since the set \( D \setminus \cup_{j=1}^N V_j \) is contained in \( \mathbb{R}^n \times [T_0 + \delta, T] \), we have
\[ e^{\tau T_0} \int_{D \setminus \cup_{j=1}^N V_j} \rho(x, t)e^{\sqrt{\tau}x \omega - \tau t}dxdt = O(e^{-\tau\delta/2}) \quad (3.8) \]
as \( \tau \to \infty \).

Here we claim as \( \tau \to \infty \)
\[ e^{\tau T_0} \int_{V_j} \rho(x, t)e^{\sqrt{\tau}x \omega - \tau t}dxdt = \tau^{-(n+1)} e^{\sqrt{\tau}x_j \omega} \left\{ \rho(x_j, T_0) \left\| W_j \right\| n! + O\left( \frac{1}{\tau^{\delta_j}} \right) \right\}. \quad (3.9) \]
This can be proved as follows. Write
\[ e^{\tau T_0} \int_{V_j} \rho(x, t)e^{\sqrt{\tau}x \omega - \tau t}dxdt \]
\[ = e^{\tau T_0} \int_0^\delta \left( \frac{s}{\delta} \right)^n ds \int_{W_j} \rho(x_j + \frac{s}{\delta} (y - x_j), T_0 + s) e^{\sqrt{\tau}(x_j + \frac{1}{\delta}(y - x_j)) \omega} e^{-\tau(T_0 + s)}dy \]
\[ = e^{\sqrt{\tau}x_j \omega} \int_0^\delta \left( \frac{s}{\delta} \right)^n e^{-\tau s} ds \int_{W_j} \rho(x_j + \frac{s}{\delta} (y - x_j), T_0 + s) e^{\sqrt{\tau}(y - x_j) \omega} dy \]
\[ \equiv I + II \]
where
\[ I = e^{\sqrt{\tau x_j} \omega} \rho(x_j, T_0) \int_0^{\delta} \left( \frac{s}{\delta} \right)^n e^{-\tau s} ds \int_{W_j} e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega} dy \]
and
\[ II = e^{\sqrt{\tau x_j} \omega} \int_0^{\delta} \left( \frac{s}{\delta} \right)^n e^{-\tau s} ds \int_{W_j} \{ \rho(x_j + \frac{s}{\delta}(y - x_j), T_0 + s) - \rho(x_j, T_0) \} e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega} dy. \]

The change of variables gives
\[ I = e^{\sqrt{\tau x_j} \omega} \int_0^{\delta} \left( \frac{s}{\delta} \right)^n e^{-\tau s} ds \int_{W_j} \rho(x_j, T_0) \left( \int_0^{\delta} \left( \frac{\xi}{\delta} \right)^n e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega - \xi d\xi} \right) dy \]
and thus from Lebesgue’s dominated convergence theorem we obtain
\[ \tau^{n+1} e^{-\sqrt{\tau x_j} \omega} I \to \rho(x_j, T_0) \int_{W_j} \left( \int_0^{\delta} \left( \frac{\xi}{\delta} \right)^n e^{-\xi d\xi} \right) dy = \rho(x_j, T_0) \frac{|W_j|}{\delta^n} n! \quad (3.10) \]
as \( \tau \to \infty. \)

On the other hand, we have
\[
\begin{align*}
&e^{-\sqrt{\tau x_j} \omega} |II| \leq C \int_0^{\delta} \left( \frac{s}{\delta} \right)^n e^{-\tau s} ds \int_{W_j} \left( \frac{s}{\delta}(y - x_j) \right)^2 + s^2 \theta_j/2 e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega} dy \\
&\leq C' \int_0^{\delta} s^{n+\theta_j} e^{-\tau s} ds \int_{W_j} e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega} dy \\
&= \frac{C'}{\tau^{n+\theta_j}} \int_0^{\tau^\delta} \xi^{n+\theta_j} e^{-\xi} d\xi \int_{W_j} e^{\frac{\sqrt{\tau}}{\tau}(y - x_j) \omega} dy \\
&= O\left( \frac{1}{\tau^{n+1+\theta_j}} \right)
\end{align*}
\]
as \( \tau \to \infty. \)

A combination of (3.10) and (3.11) yields the desired conclusion (3.9).

Now from (3.9) we have
\[
\begin{align*}
e^{\tau T_0} \sum_{j=1}^{N} \int_{V_j} \rho(x, t) e^{\sqrt{\tau x_j} \omega - \tau t} dx dt \\
= \tau^{-(n+1)} e^{\sqrt{\tau} \max_{j=1, \ldots, N} x_j \omega} \left( \sum_{x_i \omega = \max_{j=1, \ldots, N} x_j \omega} \rho(x_i, T_0) \frac{|W_i|}{\delta^n} n! + O\left( \frac{1}{\tau^{\theta_i}} \right) \right) \\
+ \sum_{x_i \omega < \max_{j=1, \ldots, N} x_j \omega} e^{-\sqrt{\tau}(\max_{j=1, \ldots, N} x_j \omega - x_i \omega)} \rho(x_i, T_0) \frac{|W_i|}{\delta^n} n! + O\left( \frac{1}{\tau^{\theta_i}} \right) \\
= \tau^{-(n+1)} e^{\sqrt{\tau} \max_{j=1, \ldots, N} x_j \omega} \left( \sum_{x_i \omega = \max_{j=1, \ldots, N} x_j \omega} \rho(x_i, T_0) \frac{|W_i|}{\delta^n} n! + O\left( \frac{1}{\tau^{\theta_i}} \right) \right)
\end{align*}
\]

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as \( \tau \to \infty \) where \( \theta_0 \) is the number given by (3.4). Then, from (3.5)\textendash(3.8) and (3.12) we obtain, as \( \tau \to \infty \)

\[
I_{\omega}(\tau; T_0) = \tau^{-(n+1)}e^{\sqrt{\tau} \max_{j=1,\ldots,N} x_j \omega} \left\{ \sum_{x_i, \omega=\max_{j=1,\ldots,N} x_j \omega} \rho(x_i, T_0) \frac{|W_i|}{\delta^n} n! + O\left(\frac{1}{\tau^{\theta_0}}\right) \right\}.
\]

Then every statement in Theorem 3.2 immediately follow from this asymptotic formula and the last assumption on \( f \).

\( \square \)

**Remark 3.1.** The method works also for the equation

\[
u_t = \Delta u - V \cdot \nabla u + f(x, t)
\]

where \( V = (V_1, \ldots, V_n) \) is a constant vector. In this case the function

\[
\tilde{v}(x, t) = e^{(\sqrt{\tau} - \frac{\nu}{2})x} e^{(\frac{\nu V}{4} - \tau)t}
\]
satisfies the backward equation

\[
v_t + \Delta v + V \cdot \nabla v = 0
\]
in the whole space. Define the indicator function by the formula

\[
J_{\omega}(\tau, s) = e^{s \tau} \int_0^T \int_{\partial \Omega} \left( \frac{\partial \tilde{v}}{\partial \nu} u - \frac{\partial u}{\partial \nu} \tilde{v} + (V \cdot \nu) u \tilde{v} \right) dSdt, \ \tau > 0.
\]

Then we have the same formula as (3.3) under the same assumptions on \( f \) except for replacing the last condition on \( f \) mentioned above with the condition

\[
\sum_{x_i, \omega=\max_{j=1,\ldots,N} x_j \omega} \rho(x_i, T_0) e^{-\frac{\nu}{2} x_i |W_i|} \neq 0
\]

and \( \rho(x_i, T_0) \) in the forth term of (3.3) with \( \rho(x_i, T_0) e^{-\frac{\nu}{2} x_i e^{\frac{\nu V}{4} T_0}} \).

The same idea works also for another inverse source problem. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n(n = 2, 3) \) with smooth boundary. Let \( T \) be an arbitrary positive number. Let \( u = u(x, t) \) satisfy

\[
u_t = \Delta u \text{ in } \Omega \times [0, T[.
\]

**Inverse Problem 3.2.** Extract the *initial temperature* \( u(\cdot, 0) \) from the data \( u|_{\partial \Omega \times [0, T]}, \ \partial u/\partial \nu|_{\partial \Omega \times [0, T]} \).

This is also a classical and typical ill-posed problem. Here assuming the special form on \( u(x, 0) \), we give an extraction formula of the information about the *discontinuity* of \( u(x, 0) \).

We define another indicator function by using real exponential solutions of the backward heat equation instead of complex exponential solutions. Given \( \omega \in S^{n-1} \) define the *indicator function* \( I_{\omega}(\tau) \) by the formula

\[
I_{\omega}(\tau) = \int_0^T \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) dSdt, \ \tau > 0
\]

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where
\[ \nu(x, t) = e^{\sqrt{\tau} x \cdot \omega - \tau t}. \]

We assume that the initial temperature \( u(x, 0) \) takes the form
\[ u(x, 0) = \chi_D(x) \rho(x) \]
where \( D \) is an unknown open subset of \( \Omega \); \( \rho \) is an unknown essentially bounded function on \( D \).

We say that \( \rho \) has a **jump** at the direction \( \omega \in S^{n-1} \) if there exist positive constants \( C = C(\omega) \) and \( \delta = \delta(\omega) \) such that \( \rho(x) \geq C \) for almost all \( x \in D_\omega(\delta) \) or \( -\rho(x) \geq C \) for almost all \( x \in D_\omega(\delta) \) where
\[ D_\omega(\delta) = \{ x \in D \mid h_D(\omega) - \delta < x \cdot \omega < h_D(\omega) \}. \]

We say that \( D \) is **regular** at the direction \( \omega \) if there exist positive constants \( M = M(\omega) \) and \( \epsilon = \epsilon(\omega) \) and \( p \in [0, \infty[ \) such that for all \( s \in ]0, \epsilon[ \)
\[ \mu_{n-1}(\{ x \in D \mid x \cdot \omega = h_D(\omega) - s \}) \geq Ms^p. \]

For example, if \( \partial D \) is Lipschitz, then \( D \) is regular at all directions.

These conditions have been appeared already in [29] where inverse boundary value problems for elliptic equations were considered.

Here we state the following simple fact.

**Theorem 3.3.** Let \( \omega \in S^{n-1} \). Assume that \( \rho \) has a jump at direction \( \omega \) and \( D \) is regular at \( \omega \). Then the formula
\[ \lim_{\tau \to \infty} \frac{\log |I_\omega(\tau)|}{\sqrt{\tau}} = h_D(\omega), \tag{3.13} \]
is valid.

**Proof.** Integration by parts yields
\[ I_\omega(\tau) = \int_D \rho(x)e^{\sqrt{\tau} x \cdot \omega} dx - e^{-\tau T} \int_\Omega u(x, T)e^{\sqrt{\tau} x \cdot \omega} dx. \tag{3.14} \]

Write
\[ \int_D \rho(x)e^{\sqrt{\tau} x \cdot \omega} dx = e^{\sqrt{\tau} h_D(\omega)} \int_D \rho(x)e^{-\sqrt{\tau}(h_D(\omega) - x \cdot \omega)} dx. \]

Then, using a similar argument in [29], we see that, for all \( \tau \geq \tau_0 \)
\[ C_1 e^{\sqrt{\tau} h_D(\omega)} (\sqrt{\tau})^{-(p+1)} \leq \int_D \rho(x)e^{\sqrt{\tau} x \cdot \omega} dx \leq C_2 e^{\sqrt{\tau} h_D(\omega)} \]
where \( \tau_0, C_1 \) and \( C_2 \) are positive constants. The second term of (3.14) has the estimation \( O(e^{\sqrt{\tau} h_D(\omega)} e^{-\tau T/2}) \) as \( \tau \to \infty \). Then we immediately obtain the formula (3.13). \( \Box \)
3.3 Extracting Unknown Boundary/Interface and Virtual Signal

In this subsection we consider two typical inverse initial boundary problems for the heat equation in one space dimension and explain about a role of complex exponential solutions for the backward heat equation.

Let us describe the first problem. Let $a > 0$ and $\rho \geq 0$. Let $u = u(x, t)$ be an arbitrary solution of the problem:

\[
\begin{align*}
    u_t &= u_{xx} \text{ in }]0, a[ \times \]0, T[, \\
    u_x(a, t) + \rho u(a, t) &= 0 \text{ for } t \in ]0, T[, \\
    u(x, 0) &= 0 \text{ in }]0, a[.}
\end{align*}
\]

**Inverse Problem 3.3.** Assume that both $a$ and $\rho$ are unknown. Extract $a$ from $u(0, t)$ and $u_x(0, t)$ for $0 < t < T$.

There are extensive studies for the uniqueness and stability issue for the corresponding problem in multi space dimensions. See [6, 11, 81, 82] and references therein.

In one space dimensional case, in general, it is possible to calculate so-called the response operator for the heat equation from the data above. Therefore it may be possible to apply the method in [1] to the response operator and get the spectral data. Thus the problem may be reduced to the inverse spectral problem which has been well studied. However, this is due to the special situation of one space dimension and the procedure to get the spectral data is not an easy way.

In this subsection, without reducing to other inverse problems, we present an independent and simpler approach which is an application of the enclosure method to Problem 3.3. Define

\[
v(x, t) = e^{-z^2 t} e^{xz}
\]

where $\tau > c^{-2}$ and $z$ is given by

\[
z = -c\tau \left( 1 + i \sqrt{1 - \frac{1}{c^2 \tau}} \right).
\]

The function $v$ satisfies the backward heat equation $\partial_t v + \Delta v = 0$; $z$ satisfies the equation

\[
z^2 = \tau + 2 c^2 \tau^2 \sqrt{1 - \frac{1}{c^2 \tau}}.
\]

For each $s$ we have $|e^{s} v(x, t)| = e^{r(s-t-cx)}$. This yields that the asymptotic behaviour of $e^{s} v(x, t)$ as $\tau \to \infty$ divides the space time into two parts $s-t-cx > 0$ and $s-t-cx < 0$: if $s-t-cx > 0$, then $|e^{s} v(x, t)|$ is growing; if $s-t-cx < 0$, then $|e^{s} v(x, t)|$ is decaying.

Define the indicator function of independent variable $\tau (> c^{-2})$ by the formula

\[
I_c(\tau; s) = e^{rs} \int_0^T (-v_x(0, t) u(0, t) + u_x(0, t) v(0, t)) dt.
\]
**Definition 3.1. Condition c** Let $c > 0$. We say that a function $f \in L^2(0, T)$ satisfies the condition $c$ if there exist positive constants $C$ and real numbers $\tau_0(\geq c^{-2})$, $\mu$ such that, for all $\tau \geq \tau_0$

$$C\tau^\mu \leq \left| \int_0^T f(t)e^{-z^2 t}dt \right|$$

where $z$ is given by (3.15).

Concerning with the condition $c$, we give two remarks:

- if $f(t)$ is given by a polynomial of $t$ on $]0, T[$ that is not identically zero, then for all $c > 0$ $f(t)$ satisfies the condition $c$
- if $f(t)$ is smooth on $[0, T[$ and $t = 0$ is not a zero point of infinite order, then for all $c > 0$ $f(t)$ satisfies the condition $c$.

And also it is easy to see that, for example, if $f(t) = e^{-1/t}$ a.e. in $0 < t < T$, then $\left| \int_0^T f(t)e^{-z^2 t}dt \right|$ is rapidly decreasing as $\tau \to \infty$. This means that this $f$ does not satisfy the condition $c$ for all $c$. Note that this case $t = 0$ is a zero point of infinite order in the sense $f(t) = O(t^m)$ for all $m = 0, 1, \cdots$ as $t \to 0$.

**Theorem 3.4 ([53]).** Assume that we know a positive number $M$ such that $M \geq 2a$. Let $c$ be an arbitrary positive number satisfying $Mc < T$. Let the $u_x(0, t)$ satisfy the condition $c$. Then the formula

$$\lim_{\tau \to \infty} \frac{\log |I_c(\tau; 0)|}{\tau} = -2ca,$$

is valid.

Once $a$ is known, then one can explicitly extract also $\rho$ from the data (Remark 4.1 in [53]).

Needless to say, from this theorem one automatically obtains the uniqueness theorem: $u(0, t)$ and $u_x(0, t)$ for $0 < t < T$ uniquely determine $a$ (and $\rho$) provided $u_x(0, t)$ satisfies the condition $c$. This type uniqueness proof is completely different from existing one. See [6] for comparison.

Here we consider: what is $2ca$?

- In the case when $\rho = 0$, $u(x, t)$ can be extended to the domain $]0, 2a[ \times ]0, T[$ as a solution of the heat equation by the reflection $x \mapsto 2a - x$ at $x = a$. In this case we think that the enclosure method yielded the information about the location of the set $]2a, \infty[ \times ]0, T[$ and $-2ca$ has the meaning

$$-2ca = \sup \left\{ \left( \begin{array}{c} x \\ t \end{array} \right) \cdot \left( \begin{array}{c} -c \\ -1 \end{array} \right) \mid (x, t) \in ]2a, \infty[ \times ]0, T[ \right\}.$$

This is nothing but the information about the value of the support function for $]2a, \infty[ \times ]0, T[$ at the direction $(-c, -1)^T$.

A more attractive interpretation is the following.

- $2ca$ is the travel time of a virtual signal with propagation speed $1/c$ that starts at the known boundary $x = 0$ and the initial time $t = 0$, reflects at another unknown boundary $x = a$ and returns to the original boundary.

The idea behind this interpretation is the belief that, in an appropriate sense

Solution of **Heat Equation** $v_t = v_{xx} \sim \sum_{\xi > 0}$ Solution of **Wave Equation** $u_t = \xi^2 u_{xx}$.
It is well known that, at least, for the initial value problem for the heat equation there is a relationship with a corresponding initial value problem for the wave equation with arbitrary fixed propagation speed (see, for example [70, 71]). So if this is true in the present case, then it is reasonable to expect that the observation date coming from the heat equation should contain some information coming from the wave equation with arbitrary propagation speed $\xi = 1/c$. This is the role of $c$. This suggests that the indicator function for the heat equation is a mathematical instrument that picks up a signal coming from the corresponding wave equation with propagation speed $1/c$. Of course, to get $a$ only from the observation data one can just use a small single $c$ with $Mc < T$.

Note that, in [54] we have already confirmed that this interpretation works also for the case when the heat conductivity of the material is given by a smooth function or piecewise constant function.

The second problem is the following.

Let $0 < b < a$. Define

$$
\gamma(x) = \begin{cases} 
\gamma_1, & \text{if } 0 < x < b, \\
\gamma_2, & \text{if } b < x < a
\end{cases}
$$

where both $\gamma_1$ and $\gamma_2$ are positive constants and satisfies $\gamma_2 \neq \gamma_1$.

Let $u$ be an arbitrary solution of the problem:

$$u_t = (\gamma u_x)_x \text{ in }]0, a[ \times ]0, T[,$$

$$u(x, 0) = 0 \text{ in }]0, a[. $$

**Inverse Problem 3.4.** Assume that $\gamma_1$ is known and that $a$, $b$ and $\gamma_2$ are all unknown. Extract $b$ from $u(0, t)$ and $\gamma_1 u_x(0, t)$ for $0 < t < T$.

Let $c$ be an arbitrary positive number. Let

$$v(x, t) = e^{-z^2t}e^{xz/\sqrt{\gamma_1}}$$

where $z$ is given by (3.15).

The function $v$ satisfies the backward heat equation $v_t + \gamma_1 v_{xx} = 0$ in the whole space-time.

Define the indicator function of independent variable $\tau(> c^{-2})$ by the formula

$$I_c(\tau) = \int_0^T (-\gamma_1 v_x(0, t)u(0, t) + \gamma_1 u_x(0, t)v(0, t)) dt.$$ 

Note that this indicator function makes use of the observation data at $x = 0$ only.

The following result suggests a difficulty of the “real” local problem without the control of unknown input or output at an inaccessible part.

**Theorem 3.5 ([54]).** Assume that we know a positive number $M$ such that $M \geq b/\sqrt{\gamma_1}$. Let $c$ satisfy $Mc < T$. Assume that $u_x(a, t)$ is stronger than $u_x(a, t)$ in the sense

$$
\lim_{\tau \to \infty} \frac{\int_0^T e^{-z^2t}u_x(a, t)dt}{\int_0^T e^{-z^2t}u_x(0, t)dt} \exp \left( c \tau \left( \frac{b}{\sqrt{\gamma_1}} - \frac{(a - b)}{\sqrt{\gamma_2}} \right) \right) = 0 \quad (3.16)
$$

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and that \( u_x(0, t) \) satisfies the condition \( c \). Then the formula

\[
\lim_{\tau \to -\infty} \frac{\log |I_c(\tau)|}{\tau} = -2 \frac{cb}{\sqrt{\gamma_1}},
\]

is valid.

Needless to say, this theorem also automatically yields a uniqueness theorem.

Note that \( 2cb/\sqrt{\gamma_1} \) is the \textit{travel time} of a virtual signal with propagation speed \( \sqrt{\gamma_1}/c \) that starts at the known boundary \( x = 0 \) and the initial time \( t = 0 \), reflects at another unknown boundary \( x = b \) and returns to the original boundary. The condition (3.16) plays the role of killing another virtual signal with propagation speed \( \sqrt{\gamma_2}/c \) that starts at the unknown boundary \( x = a \) and the initial time \( t = 0 \), refracts at another unknown boundary \( x = b \) and goes to the boundary \( x = 0 \).

Applying these ideas developed in this section to similar inverse problems in two and three space dimensional cases is of great interest. These occupy the central parts of our next projects. For more information about these inverse problems including conjectures see [56].

4 Probe Method and A Carleman Function

4.1 Probe Method

In [51] we applied the \textit{probe method} to Inverse Problem 2.1(see subsection 2.1 for notation). The probe method is a method of \textit{probing inside} given material by monitoring the behaviour of the sequence of the energy gap

\[
\int_{\partial \Omega} \left\{ (\Lambda_0 - \Lambda_\Sigma)(v_n|_{\partial \Omega}) \right\} \mathbb{I}_n|_{\partial \Omega} dS
\]

for a specially chosen sequence \( \{v_n\} \) of solutions of the governing equation for the background medium \( \Sigma = \emptyset \) which play a role of \textit{probe needle}.

The method starts with introducing

Definition 4.1. Given a point \( x \in \Omega \) we say that a non self-intersecting piecewise linear curve \( \sigma \) in \( \overline{\Omega} \) is a needle with tip at \( x \) if \( \sigma \) connects a point on \( \partial \Omega \) with \( x \) and other points of \( \sigma \) are contained in \( \Omega \). We denote by \( N_x \) the set of all needles with tip at \( x \).

We call \( \sigma \in N_x \) a \textit{needle with tip at} \( x \).

We insert this needle into the body \( \Omega \) like as a past dentist does, however, virtually. The meaning of “probe” in the probe method is coming from the use of this virtual needle.

Let \( b \) be a nonzero vector in \( \mathbb{R}^m \). Given \( x \in \mathbb{R}^m \), \( \rho > 0 \) and \( \theta \in [0, \pi] \) set

\[
C_x(b, \theta/2) = \{ y \in \mathbb{R}^m \mid (y - x) \cdot b > |y - x||b| \cos(\theta/2) \}
\]

and

\[
B_\rho(x) = \{ y \in \mathbb{R}^m \mid |y - x| < \rho \}.
\]

A set having the form

\[
V = B_\rho(x) \cap C_x(b, \theta/2)
\]

for some \( \rho, b, \theta \) and \( x \) is called a \textit{finite cone} with \textit{vertex} at \( x \).
Let \( G(y) \) be a solution of the Laplace equation in \( \mathbb{R}^m \setminus \{0\} \) such that, for any finite cone \( V \) with vertex at 0
\[
\int_V |\nabla G(y)|^2 \, dy = \infty. \tag{4.1}
\]

Hereafter we fix this \( G \).

For the new formulation of the probe method we need the following.

**Definition 4.2.** Let \( \sigma \in \mathcal{N}_x \). We call the sequence \( \xi = \{v_n\} \) of \( H^1(\Omega) \) solutions of the Laplace equation a **needle sequence** for \( (x, \sigma) \) if it satisfies, for each fixed compact set \( K \subset \Omega \setminus \sigma([0, 1]) \)
\[
\lim_{n \to \infty} (|v_n(\cdot) - G(\cdot - x)|_{L^2(K)} + \|\nabla\{v_n(\cdot) - G(\cdot - x)\}\|_{L^2(K)}) = 0.
\]

Needless to say, the existence of the needle sequence is a consequence of the Runge approximation property of the Laplace equation.

In [49] we clarified the behaviour of the needle sequence on the needle as \( n \to \infty \).

The two lemmas given below are the core of the new formulation of the probe method (see [49] for the proof).

**Lemma 4.1.** Let \( x \in \Omega \) be an arbitrary point and \( \sigma \) be a needle with tip at \( x \). Let \( \xi = \{v_n\} \) be an arbitrary needle sequence for \( (x, \sigma) \). Then, for any finite cone \( V \) with vertex at \( x \) we have
\[
\lim_{n \to \infty} \int_{V \cap \Omega} |\nabla v_n(y)|^2 \, dy = \infty.
\]

**Lemma 4.2.** Let \( x \in \Omega \) be an arbitrary point and \( \sigma \) be a needle with tip at \( x \). Let \( \xi = \{v_n\} \) be an arbitrary needle sequence for \( (x, \sigma) \). Then for any point \( z \in \sigma([0, 1]) \) and open ball \( B \) centered at \( z \) we have
\[
\lim_{n \to \infty} \int_{B \cap \Omega} |\nabla v_n(y)|^2 \, dy = \infty.
\]

From these lemmas we know that one can recover **full knowledge** of the given needle as the set of all points where the needle sequence for the needle blows up. This means that the needle is realized as a special sequence of harmonic functions without losing information about the geometry of the needle. This is the new point added on the probe method.

**Definition 4.3.** Given \( x \in \Omega \), needle \( \sigma \) with tip \( x \) and needle sequence \( \xi = \{v_n\} \) for \( (x, \sigma) \) define
\[
I(x, \sigma, \xi)_n = \int_{\partial \Omega} \{(\Lambda_0 - \Lambda_\Sigma) f_n\} \, dS, \ n = 1, 2, \ldots
\]
where
\[
f_n(y) = v_n(y), \ y \in \partial \Omega.
\]
\( \{I(x, \sigma, \xi)_n\}_{n=1,2,\ldots} \) is a sequence depending on \( \xi \) and \( \sigma \in \mathcal{N}_x \). We call the sequence the **indicator sequence**.

Note that this definition gives an explanation of inserting a needle into the body virtually.

The procedure is consists of two steps:

1. given \( x \in \Omega \) prepare a needle \( \sigma \) with tip at \( x \);
2. apply the voltage potential on the surface of the body that is the trace of each term of a needle sequence for \( (x, \sigma) \) onto the surface.
The behaviour of the indicator sequence has two sides. One side is closely related to the function defined below.

**Definition 4.4.** The *indicator function* \( I \) is defined by the formula

\[
I(x) = \int_{\Omega \setminus \Sigma} \left| \nabla w_x \right|^2 dy, \quad x \in \Omega \setminus \Sigma
\]

where \( w_x \) is the unique weak solution of the problem:

\[
\begin{align*}
\Delta w &= 0 \text{ in } \Omega \setminus \Sigma, \\
\frac{\partial w}{\partial \nu} &= -\frac{\partial}{\partial \nu} (G(\cdot - x)) \text{ on } \Sigma, \\
w &= 0 \text{ on } \partial \Omega.
\end{align*}
\]

The function \( w_x \) is called the *reflected solution* by \( \Sigma \).

The following theorem describes the important properties of the indicator function and gives a way of calculating the value by using the indicator sequence for a suitable needle.

**Theorem 4.3.A ([51]).** We have:

1. **(A.1)** given \( x \in \Omega \setminus \Sigma \) and needle \( \sigma \) with tip at \( x \) if \( \sigma([0, 1]) \cap \Sigma = \emptyset \), then for any needle sequence \( \xi = \{v_n\} \) for \( (x, \sigma) \) the sequence \( \{I(x, \sigma, \xi)_n\} \) converges to the indicator function \( I(x) \);
2. **(A.2)** for each \( \varepsilon > 0 \)

\[
\sup_{\text{dist}(x, \Sigma) > \varepsilon} I(x) < \infty;
\]
3. **(A.3)** given point \( a \in \text{Int} \Sigma \)

\[
\lim_{x \to a} I(x) = \infty.
\]

Note that, in the case when \( \text{Int} \Sigma \) is smooth, Theorem 4.3.A has been established in [60].

The following theorem gives an answer to the natural question: *what happens on the indicator sequence when the tip of the needle is just located on the crack or passing through the crack?*

In the previous applications of the probe method [60, 78] this type problem was not considered and it is difficult to apply the techniques developed in the papers to the problem. However, in [51] we found a completely different technique which gives the proof of Theorem 4.3.A and the following theorem at the same time.

**Theorem 4.3.B ([51]).** Let \( x \in \Omega \setminus \partial \Sigma \) and \( \sigma \in N_x \) satisfy \( \emptyset \neq \sigma([0, 1]) \cap \Sigma \subset \text{Int} \Sigma \). Then for any needle sequence \( \xi = \{v_n\} \) for \( (x, \sigma) \) we have \( \lim_{n \to \infty} I(x, \sigma, \xi)_n = \infty \).

This is the essence of the probe method applied to an inverse crack problem. In this section we consider *how to construct* the needle sequence in the probe method explicitly in a closed-form.

There are several points for the meaning of the “construction”. It is well known that one can construct the needle sequence by solving infinitely many first kind integral equations in the sense of minimum norm solutions. The minimum norm solutions are given by a combination of the Tikhonov regularization method and Morozov discrepancy principle (see
for these concepts). However, from our point of view this does not yield an explicit needle sequence. One can also point out that the determination of the regularization parameter via the Morozov discrepancy principle itself is a nonlinear problem and seems impossible to find the parameter explicitly.

It should be pointed out that the idea of the multi pole expansion also yields an explicit needle sequence for a straight needle (see Definition 4.5). However the needle sequence does not have a closed-form (see [59]).

4.2 Yarmukhamedov’s Fundamental Solution and A Standard Needle Sequence

So how can one construct the needle sequence explicitly? In this and next subsection we consider only the case when $m = 3$.

The idea came from reconsidering the role of a Carleman function which is a special fundamental solution with a large parameter for the Laplace operator and gives a formula to calculate the value of the solution of the Cauchy problem in a domain for the Laplace equation. Roughly speaking, we say that a function $\Phi(y, x, \tau)$ depending on a large parameter $\tau > 0$ is called a Carleman function for a domain $\Omega$ and the portion $\Gamma$ of $\partial \Omega$ if, it satisfies the equation $\Delta u + \delta(y - x) = 0$ in $\Omega$; for each fixed $x \in \Omega$ $\Phi(\cdot, x, \tau)$ and $\partial/\partial \nu \Phi(\cdot, x, \tau)$ on $\partial \Omega \setminus \Gamma$ vanish as $\tau \to \infty$. This function gives a representation formula of any solution of the Cauchy problem for the Laplace equation in $\Omega$ by using only the Cauchy data on $\Gamma$ and yields a natural regularization for the numerical computation of the solution from a noisy inaccurate Cauchy data on $\Gamma$.

The Cauchy problem for the Laplace equation is a fundamental and important ill-posed problem appearing in mathematical sciences, engineering and medicine. Therefore it is quite important to seek an explicit Carleman function for several domains. For this problem Yarmukhamedov [87] gave a very interesting Carleman function for a special domain in three dimensions which is a special version of his fundamental solutions [86](see also [85]) for the Laplace operator.

**Theorem 4.4 ([86]).** Let $\lambda \geq 0$. Let $K(w)$ be an entire function such that
- $K(w)$ is real for real $w$
- $K(0) = 1$
- for each $R > 0$ and $m = 0, 1, 2$

$$\sup_{|Re w| < R} |K^{(m)}(w)| e^{\lambda |Im w|} < \infty.$$ (4.2)

Define

$$-2\pi^2 \Phi(x) = \frac{1}{2} \int_0^\infty \text{Im} \left( \frac{K(w)}{w} \right) \frac{e^{\lambda u} + e^{-\lambda u}}{\sqrt{|x'|^2 + u^2}} du.$$ (4.3)

where $w = x_3 + i \sqrt{|x'|^2 + u^2}$ and $x' = (x_1, x_2)$. Then one has the expression

$$\Phi(x) = \frac{e^{\lambda |x|}}{4\pi |x|} + H(x)$$
where $H$ is $C^2$ in the whole space and satisfies
\[ \triangle H(x) + \lambda^2 H(x) = 0 \text{ in } \mathbb{R}^3 \]
and therefore $\Phi$ satisfies
\[ \triangle \Phi(x) + \lambda^2 \Phi(x) + \delta(x) = 0 \text{ in } \mathbb{R}^3. \]

Hereafter we set $\Phi = \Phi_K$ to denote the dependence on $K$ and consider only the case when $\lambda = 0$. In this case (2.2) ensures that $K(w) = E_{\alpha}(\tau w)$ with $\tau > 0$ satisfies (4.2). Yarmukhamedov [87] established that the function $\Phi_K$ for this $K$ with a fixed $\alpha$ is a Carleman function (for the Laplace equation).

In this and following subsections we always consider a geometrically simplest needle described in

**Definition 4.5.** A needle with tip at $x$ is called a *straight needle* with tip at $x$ directed to $\omega$ if the needle is given by $l_x(\omega) \cap \Omega$ where
\[
l_x(\omega) = \{x + t\omega \mid 0 \leq t < \infty\}.\]

**Definition 4.6.** Given two unit vectors $\vartheta_1$ and $\vartheta_2$ in three dimensions and $\alpha \in [0, 1]$ define
\[
v(y; \alpha, \tau, \vartheta_1, \vartheta_2) = -\left\{ \Phi_K(y \cdot \vartheta_1, y \cdot \vartheta_2, y \cdot (\vartheta_1 \times \vartheta_2)) - \frac{1}{4\pi |y|} \right\}, \tau > 0\]
where $K(w) = E_{\alpha}(\tau w)$. From Theorem 4.4 for $\lambda = 0$ one knows that this function of $y$ is nothing but the regular part of $\Phi_K$ and harmonic in the whole space.

The result below can be proved along the line of that of Lemma in [87].

**Theorem 4.5 ([55]).** Let $x \in \Omega$ and $\sigma$ be a straight needle with tip at $x$ directed to $\omega = \vartheta_1 \times \vartheta_2$. Then the sequence $\{v(\cdot - x; \alpha_n, \tau_n, \vartheta_1, \vartheta_2)|_{\Omega}\}$ is a needle sequence for $(x, \sigma)$ with
\[
G(y) = \frac{1}{4\pi|y|}\]
where $\alpha_n$ and $\tau_n$ are suitably chosen sequences and satisfy
- $0 < \alpha_n < 1, \tau_n > 0$
- $\alpha_n \to 0$ and $\tau_n \to \infty$.

Note that this $G$ satisfies (4.1) for any finite cone $V$ with vertex at 0.

The key point of the proof of Theorem 4.5 is Mittag-Leffler’s integral representation (p. 206 of [4]).

**Definition 4.7.** We call the needle sequence $\{v(\cdot - x; \alpha_n, \tau_n, \vartheta_1, \vartheta_2)|_{\Omega}\}$ given in Theorem 4.5 a *standard* needle sequence for $(x, \sigma)$ for the Laplace equation.

Choosing $K(w) = 1$ in $\Phi_K$, from (4.3) we have
\[
\Phi_1(y) = \frac{1}{4\pi|y|}.
\]

This gives the expression:
\[
2\pi^2 v(y; \alpha, \tau, \vartheta_1, \vartheta_2) = \int_0^\infty \Im \left( \frac{E_{\alpha}(\tau w) - 1}{w} \right) \frac{du}{\sqrt{|y \cdot \vartheta_1|^2 + |y \cdot \vartheta_2|^2 + u^2}}
\]
where
\[ w = y \cdot \omega + i \sqrt{|y \cdot \partial_1|^2 + |y \cdot \partial_2|^2 + u^2}. \]

The advantage of the standard needle sequence is that one can exactly know the precise values in a closed-form on the needle and its linear extension.

**Theorem 4.6 ([55]).** Let \( \omega = \vartheta_1 \times \vartheta_2 \). We have:

1. if \( y = x + s \omega \) with \( s \neq 0 \), then
   \[
   v(y - x; \alpha, \tau, \vartheta_1, \vartheta_2) = \frac{E_\alpha(\tau s) - 1}{4\pi s} \omega.
   \]
   \[
   \nabla v(y - x; \alpha, \tau, \vartheta_1, \vartheta_2) = \frac{d}{ds} \left\{ \frac{E_\alpha(\tau s) - 1}{4\pi s} \right\} \omega.
   \]

2. if \( y = x \), then
   \[
   v(y - x; \alpha, \tau, \vartheta_1, \vartheta_2)|_{y=x} = \frac{\tau}{4\pi \Gamma(1 + \alpha)} \omega.
   \]
   \[
   \nabla v(y - x; \alpha, \tau, \vartheta_1, \vartheta_2)|_{y=x} = \frac{\tau^2}{4\pi \Gamma(1 + 2\alpha)} \omega.
   \]

Therefore \( \nabla v \) on the line \( y = x + s \omega \) \((-\infty < s < \infty)\) is parallel to the direction of the line. In particular, using the power series expansion of the Mittag-Leffler function, we see that \( \nabla v(y - x; \alpha, \tau, \vartheta_1, \vartheta_2) \cdot \omega > 0 \) and \( \lim_{n \to \infty} \nabla v(y - x; \alpha_n, \tau_n, \vartheta_1, \vartheta_2) \cdot \omega = +\infty \) on the set \( l_x(\omega) \).

### 4.3 A Needle Sequence for The Helmholtz Equation

The existence of the needle sequence for the Helmholtz equation has been ensured under the additional condition on \( \lambda \) (see Appendixes of [25, 49] for the proof):

- \( \lambda^2 \) is not a Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \).

However, the proof therein does not give us any explicit form of a needle sequence. Since the needle sequence for the Helmholtz equation plays a central role in the probe method applied to inverse obstacle scattering problems, it is quite important to obtain an explicit one.

In this subsection we give explicit needle sequences for the Helmholtz equation for all straight needles and all \( \lambda (> 0) \). There is no additional condition on \( \lambda \). For the construction we do not make use of Theorem 4.4. The Mittag-Leffler function with \( \alpha < 1 \) does not satisfy the condition (4.2) when \( \lambda > 0 \). Therefore one can not substitute \( K(w) = E_\alpha(\tau w) \) into (4.3) to construct a needle sequence for the Helmholtz equation \( \Delta u + \lambda^2 u = 0 \) in three dimensions. In this section, instead of (4.3) we employ an idea of making use of a transformation introduced by Vekua [79, 80].

Let \( \lambda > 0 \). The Vekua transform \( v \mapsto T_\lambda v \) in three dimensions takes the form

\[
T_\lambda v(y) = v(y) - \frac{\lambda|y|}{2} \int_0^1 v(ty) J_1(\lambda|y|\sqrt{1-t}) \sqrt{\frac{t}{1-t}} dt.
\]
where \( J_1 \) stands for the Bessel function of order 1. The important property of this transform is: if \( v \) is harmonic in the whole space, then \( T_\lambda v \) is a solution of the Helmholtz equation \( \triangle u + \lambda^2 u = 0 \) in the whole space.

**Definition 4.8.** Given two unit vectors \( \vartheta_1 \) and \( \vartheta_2 \) in three dimensions and \( \alpha \in ]0, 1] \) define

\[
v^\lambda(y; \alpha, \tau, \vartheta_1, \vartheta_2) = T_\lambda v(y; \alpha, \tau, \vartheta_1, \vartheta_2), \quad \tau > 0.
\]

This function of \( y \) satisfies the Helmholtz equation \( \triangle u + \lambda^2 u = 0 \) in the whole space.

**Theorem 4.7 ([55]).** Let \( x \in \Omega \) and \( \sigma \) be a straight needle with tip at \( x \) directed to \( \omega = \vartheta_1 \times \vartheta_2 \). Then the sequence \( \{v^\lambda(\cdot - x; \alpha_n, \tau_n, \vartheta_1, \vartheta_2)|_\Omega \} \) is a needle sequence for \((x, \sigma)\) for the Helmholtz equation with \( G = G_\lambda \) given by

\[
G_\lambda(y) = \text{Re}\left(\frac{e^{i\lambda|y|}}{4\pi|y|}\right)
\]

where \( \alpha_n \) and \( \tau_n \) are suitably chosen sequences and satisfy

- \( 0 < \alpha_n < 1, \quad \tau_n > 0 \)
- \( \alpha_n \to 0 \) and \( \tau_n \to \infty \).

This \( G \) also satisfies (4.1) for any finite cone \( V \) with vertex at 0.

**Definition 4.9.** We call the needle sequence \( \{v^\lambda(\cdot - x; \alpha_n, \tau_n, \vartheta_1, \vartheta_2)|_\Omega \} \) given in Theorem 4.7 a standard needle sequence for \((x, \sigma)\) for the Helmholtz equation \( \triangle u + \lambda^2 u = 0 \).

Therefore now we can say that the probe method applied to the inverse obstacle scattering problems becomes a completely explicit method if one uses only straight needles and standard needle sequences. In this case this explicit method gives information of the location and shape of unknown obstacles more than the convex hull. It will be very interested in doing the numerical testing of the method in three dimensions.

Summing up, we can say that the central part of the probe method for discontinuity embedded in a homogeneous background medium is based on the property of the Mittag-Leffler function.

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**Acknowledgement**

This research was partially supported by Grant-in-Aid for Scientific Research (C)(2) (No. 18540160) of Japan Society for the Promotion of Science.

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